1D compressible flow with temperature dependent transport coefficients

H. K. Jenssen[*][†]   T. K. Karper[‡]

July 14, 2009

Abstract

We establish existence of global-in-time weak solutions to the one dimensional, compressible Navier-Stokes system for a viscous and heat conducting ideal polytropic gas (pressure $p = K\theta/\tau$, internal energy $e = c_v\theta$), when the viscosity μ is constant and the heat conductivity κ depends on the temperature θ according to $\kappa(\theta) = \bar{\kappa}\theta^\beta$, with $0 \le \beta < \frac{3}{2}$. This choice of degenerate transport coefficients is motivated by the kinetic theory of gasses.

Approximate solutions are generated by a semi-discrete finite element scheme. We first formulate sufficient conditions that guarantee convergence to a weak solution. The convergence proof relies on weak compactness and convexity, and it applies to the more general constitutive relations $\mu(\theta) = \bar{\mu}\theta^\alpha$, $\kappa(\theta) = \bar{\kappa}\theta^\beta$, with $\alpha \ge 0$, $0 \le \beta < 2$ ($\bar{\mu}$, $\bar{\kappa}$ constants). We then verify the sufficient conditions in the case $\alpha = 0$ and $0 \le \beta < \frac{3}{2}$. The data are assumed to be without vacuum, mass concentrations, or vanishing temperatures, and the same holds for the weak solutions.

2000 Mathematics Subject Classification: Primary 35Q30; 76N10; Secondary 65M12.

Keywords: compressible flow, transport coefficients, degenerate, finite element method, convergence, pointwise estimates.

1 Introduction

In one space dimension (1D) the Navier-Stokes system for compressible flow of a heat conducting and viscous fluid takes the following form in Lagrangian coordinates:

$$\tau_t = u_x \qquad \text{(mass conservation)} \qquad (1.1)$$

$$u_t = \left(\mu\frac{u_x}{\tau} - p\right)_x \qquad \text{(momentum balance)} \qquad (1.2)$$

$$\mathcal{E}_t + (up)_x = \left[\frac{1}{\tau}\left(\kappa\theta_x + \mu u u_x\right)\right]_x \qquad \text{(energy conservation)} \qquad (1.3)$$

Here x=Lagrangian space variable, t=time, and the primary dependent variables are specific volume τ, fluid velocity u, and temperature θ. The specific total energy $\mathcal{E} = e + \frac{1}{2}u^2$, where e is the specific internal energy. The pressure p, the internal energy e, and the transport coefficients μ (viscosity) and κ (heat conductivity), are prescribed through constitutive relations as functions of τ and θ. The thermodynamic variables are related through Gibbs' equation $de = \theta dS - pd\tau$, where S=specific entropy. See [22] for a derivation of the model.

In this article we establish existence of weak, global-in-time solutions to the field equations (1.1)-(1.2)-(1.3) for given initial and boundary data. The flow domain is a finite interval $\Omega := (0, L)$ in Lagrangian coordinates. The data we consider are standard (see Theorem 2.1 for precise regularity assumptions): τ, u, θ are prescribed initially on Ω, while u and θ satisfy homogeneous Dirichlet and Neumann conditions, respectively, on $\partial\Omega$;

$$u = 0, \quad \theta_x = 0, \quad \text{on } \partial\Omega \times (0, T). \tag{1.4}$$

Our main interest concerns the transport coefficients μ and κ. Their dependence on τ and θ will obviously influence the solutions of the field equations as well as the mathematical analysis. Even for 1D flows there is a wide gap between the models furnished by physical theories, and the models covered by a satisfactory existence theory. We focus on the case of gases for which kinetic theory provides constitutive relations and we consider only ideal, polytropic gases:

$$p = p(\theta, \tau) = K\frac{\theta}{\tau}, \qquad e = c_v\theta, \tag{1.5}$$

where K (specific gas constant) and c_v (specific heat at constant volume) are positive constants. We scale c_v to unity such that $\mathcal{E} = \theta + \frac{1}{2}u^2$.

According to the first level of approximation in kinetic theory the viscosity μ and heat conductivity κ are functions of *temperature alone*. Furthermore, the functional dependence is the same for both coefficients. See Chapman & Cowling [5] or Vincenti & Kruger [24] for a thorough discussion of these issues. If the intermolecular potential varies as r^{-a}, r=intermolecular distance, then μ and κ are both proportional to a certain power of the temperature:

$$\mu, \kappa \propto \theta^{\frac{a+4}{2a}}.$$

For Maxwellian molecules ($a = 4$) the dependence is linear, while for elastic spheres ($a \to +\infty$) the dependence is like $\sqrt{\theta}$. In any case

$$\mu = \bar{\mu}\theta^b \qquad \text{and} \qquad \kappa = \bar{\kappa}\theta^b \qquad \text{for some } b \in (\tfrac{1}{2}, +\infty), \tag{1.6}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are constants. In particular, the transport coefficients tend to zero with θ. The discrepancy mentioned above is illustrated by the fact that, beyond the regime of small and sufficiently smooth data [13], there is no global-in-time existence result currently available for the Navier-Stokes model (1.1)-(1.2)-(1.3), with constitutive relations (1.5) and (1.6).

To put our main result Theorem 2.1 in perspective let us contrast the last statement with what is known about existence of solutions to the compressible Navier-Stokes system. (For the purpose of this introduction we concentrate on 1D flows and we consider only a small selection of the very extensive literature.) The seminal work of Kazhikhov & Shelukhin [16] treats the full one-dimensional Navier-Stokes system (1.1)-(1.2)-(1.3) with (1.5) and constant transport coefficients. Building on earlier work by Nash [21], Kanel [12], and Kazhikhov [15], global existence and uniqueness of smooth (i.e. $W^{1,2}$) solutions are established in [16] for arbitrarily large and smooth data. A key ingredient in the proof is

the pointwise a priori estimates on the specific volume which guarantee that no vacuum nor concentration of mass occur.

Much effort has been invested in generalizing this approach to other cases, and in particular to models satisfying (1.6). This has proved to be challenging. Temperature dependence of the viscosity μ has turned out to be especially problematic. On the other hand, one has been able to incorporate various forms of density dependence in μ, and also temperature dependence in κ. Dafermos [7], Dafermos & Hsiao [8] considered certain classes of solid-like materials in which the viscosity and/or the heat conductivity depend on density, and where the heat conductivity may depend on temperature. However, the latter is assumed to be bounded as well as uniformly bounded away from zero. Kawohl [14] considered a gas model that incorporates real-gas effects that occur in high-temperature regimes. In [14] the viscosity depends only on density (or is constant) and it is uniformly bounded away from zero, while the thermal conductivity may depend on both density and temperature. For example, one of the assumptions in [14] is that there are constants κ_0, $\kappa_1 > 0$ such that $\kappa(\tau, \theta)$ satisfies $\kappa_0(1 + \theta^q) \le \kappa(\tau, \theta) \le \kappa_0(1 + \theta^q)$ where $q \ge 2$. This type of temperature dependence is motivated by experimental results for gases at very high temperatures, see Zel'dovich & Raizer [28]. None of these results cover the case of a degenerate heat conductivity $\kappa(\theta) = \bar{\kappa}\theta^b$.

In the case of isentropic flow a temperature dependence in the viscosity translates into a density dependence. For some representative works in this direction see [18], [19], [20], [25], [26], [27], and references therein. Finally, for the multi-dimensional Navier-Stokes equations there has recently been established various existence results where the viscosity and/or heat conductivity depends on τ or θ, see [4] and [10]. These models do not cover the case of a gas with constitutive relations (1.5) and (1.6).

Outline The overall approach in the proof of Theorem 2.1 is standard: apriori pointwise estimates on specific volume and temperature are coupled with higher-order integral estimates to provide sufficient compactness to pass to the limit in an approximation scheme. However, in both parts of the analysis the temperature dependence in the transport coefficients raises some new issues.

To generate approximate solutions we use a semi-discrete finite element scheme. This provides an easy proof of well-posedness of the scheme (Section 3) while avoiding some of the cumbersome notation of finite difference schemes. In Section 4 we formulate certain apriori bounds on the approximations and show that these are sufficient for convergence to a weak solution (Theorem 4.1). The proof employs weak convergence and convexity techniques à la Lions [17] and Feireisl [10]. The particular proof we use, working entirely in the Lagrangian frame, seems to be new. The argument applies to ideal polytropic gases with $\mu \propto \theta^\alpha$ with $\alpha \ge 0$, and $\kappa \propto \theta^\beta$ with $\beta \in [0, 2)$, thus including the "standard" case of constant transport coefficients, as well as a full range of powers predicted by kinetic theory (see (1.6)). More general constitutive relations could presumably be included at the expense of more detailed growth conditions.

In Section 5 we verify the sufficient apriori bounds in the case where $\kappa(\theta) = \bar{\kappa}\theta^\beta$, with $\beta \in [0, \frac{3}{2})$, and μ is constant. Again the result covers constant heat conductivity and a range of power laws suggested by kinetic theory. However, we have not been able to establish sufficient apriori estimates in the case where also the viscosity depends on temperature. (We point out in the proof of Lemma 6.1 where constant viscosity seems to be essential for the argument.) To establish the apriori bounds we derive both pointwise estimates and certain energy estimates. To treat the latter we follow Hoff [11] and define energy functionals that monitor certain weighted H^1-norms in the solution. At this point the tempera-

3

ture dependence in the heat conductivity requires a careful choice of functionals. On the other hand, the assumption of constant viscosity allows us to adopt a standard argument, with only minor changes, to obtain the necessary pointwise estimates on τ and θ. For completeness this part is included in Section 6.

2 Main result

It is convenient to formulate the approximation scheme for the temperature field instead of the total energy $\mathcal{E}$. Consequently we consider a weak form of the temperature equation:

$$\theta_t - \left(\frac{\kappa\theta_x}{\tau}\right)_x = \mu\frac{u_x^2}{\tau} - pu_x. \qquad (2.1)$$

In the following definition $\Omega = (0, L)$, with $0 < L < \infty$, and we assume that μ, κ, and p are given, smooth, non-negative functions of (θ, τ).

Definition 2.1 (Weak solution) *We say that (τ, u, θ) is a weak solution of the compressible Navier-Stokes system (1.1), (1.2), (2.1) on $\Omega \times [0, T)$, with boundary conditions (1.4) and initial data (τ_0, u_0, θ_0) satisfying*

$$(\tau_0, u_0, \theta_0) \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \qquad (2.2)$$

provided that $\tau \in W^{1,2}(0, T; L^2(\Omega))$, $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega))$, $\theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$,

$$\tau_t = u_x, \quad a.e \ in \ \Omega \times (0, T), \quad \tau(x, 0) = \tau_0(x), \quad a.e \ x \in \Omega, \qquad (2.3)$$

and that for all $\phi \in C_c^\infty(\Omega \times [0, T))$:

$$\int_0^T \int_\Omega u\phi_t - \left[\frac{\mu u_x}{\tau} - p\right]\phi_x \, dxdt + \int_\Omega u_0\phi_0 \, dx = 0, \qquad (2.4)$$

$$\int_0^T \int_\Omega \theta\phi_t - \frac{\kappa}{\tau}\theta_x\phi_x + \left[\frac{\mu u_x}{\tau} - p\right]u_x\phi \, dxdt + \int_\Omega \theta_0\phi_0 \, dx = 0. \qquad (2.5)$$

In addition all terms in (2.4) and (2.5) are required to be integrable.

In the remainder of the paper we restrict ourselves to the case of an ideal polytropic gas, i.e. (1.5) holds. Furthermore, for concreteness, and motivated by kinetic theory, we only consider the case where μ and κ are proportional to (possibly different) powers of θ:

$$\mu(\theta) = \bar{\mu}\theta^\alpha, \qquad \kappa(\theta) = \bar{\kappa}\theta^\beta, \qquad (2.6)$$

where $\bar{\mu}, \bar{\kappa} > 0$, and $\alpha, \beta \geq 0$ are constants. To state the main result we define the following functionals:

$$\mathcal{A}(t) = \sup_{s \in (0,t)} \int_\Omega |u_x|^2 \, dx + \int_0^t \int_\Omega |u_t|^2 \, dxdt,$$

$$\mathcal{B}(t) = \sup_{s \in (0,t)} \int_\Omega L(\theta) \, dx + \int_0^t \int_\Omega |L'(\theta^h)_x|^2 \, dxdt,$$

$$\mathcal{D}(t) = \sup_{s \in (0,t)} \frac{1}{2}\int_\Omega \frac{1}{\tau}|L'(\theta)_x|^2 \, dx + \int_0^t \int_\Omega \kappa(\theta)|\theta_t|^2 \, dxdt,$$

where

$$L(z) := \int_0^z \int_0^\xi \kappa(\eta) \, d\eta d\xi. \qquad (2.7)$$

Our main result is:

4

Theorem 2.1 *Consider the one-dimensional, compressible Navier-Stokes system (1.1), (1.2), and (2.1) for an ideal, polytropic gas (1.5). Assume that the transport coefficients μ and κ satisfy (2.6) with $\alpha = 0$ and $0 \le \beta < \frac{3}{2}$. Let the initial data satisfy*

$$(\tau_0, u_0, \theta_0) \in L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega) , \qquad (2.8)$$

and be such that

$$C_0^{-1} \le \tau_0(x) \le C_0 , \qquad C_0^{-1} \le \theta_0(x), \qquad \text{for a.a. } x \in \Omega, \qquad (2.9)$$

where $C_0 > 0$ is a constant.

Then, for any finite time $T > 0$ there exists a weak solution (τ, u, θ) on $\Omega \times [0, T]$ of (1.1), (1.2), and (2.1), with the boundary conditions (1.4) and initial data (τ_0, u_0, θ_0). Furthermore, there exists $C > 0$, depending on the parameters K, c_v, $\bar{\mu}$, $\bar{\kappa}$, β, the initial data, and T, such that

$$C^{-1} \le \tau(x,t) \le C , \qquad \theta(x,t) \ge C^{-1} , \qquad \text{a.e in } \Omega \times (0, T), \qquad (2.10)$$

and

$$\sup_{t \in (0,T)} \left[\mathcal{A}(t) + \mathcal{B}(t) + \mathcal{D}(t) \right] \le C . \qquad (2.11)$$

The weak solution can be obtained as the pointwise a.e. limit in $\Omega \times [0, T]$ of solutions to the semi-discrete finite element scheme described in Definition 3.1.

The proof is detailed in the following sections and summarized in Section 5.1.

Remark 2.1 *Concerning the regularity of the initial data we note that the result in [11] covers discontinuous data under the assumption of BV regularity of τ_0 and u_0. The scheme we consider (Definition 3.1) is well-defined for $(\tau_0, u_0, \theta_0) \in \left[L^2(\Omega) \right]^3$; the higher regularity in (2.8) is required to bound the initial values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$. We also mention that (2.10) may be refined to give time-independent density bounds by adopting the techniques in [6].*

3 Finite element scheme

We define a semi-discrete finite element scheme approximating the compressible Navier-Stokes system. We verify the basic bounds for mass, energy and entropy for the scheme, and show that the scheme is well-defined globally in time.

3.1 Approximation scheme

Let $\{\mathbb{E}_h\}_{h>0}$ be a family of uniform meshes of Ω, where h is the mesh size and it is assumed that $N = \frac{|\Omega|}{h} \in \mathbb{N}$. For each such h, the vertices of $\mathbb{E}_h$ are $x_i = ih$, $i = 0, \ldots, N$. For each interval $E \in \mathbb{E}_h$, $\mathbb{P}_k(E)$ denotes the space of polynomials of maximal order k on E. On $\mathbb{E}_h$ we define the space of piecewise constants,

$$Q_h(\Omega) = \left\{ \phi \in L^2(\Omega); \phi|_E \in \mathbb{P}_0(E), \ \forall E \in \mathbb{E}_h \right\} ,$$

and the space of continuous piecewise linears,

$$V_h(\Omega) = \{ v \in C(\Omega); v|_E \in \mathbb{P}_1(E), \ \forall E \in \mathbb{E}_h \} .$$

Elements in $Q_h(\Omega)$ are taken to be continuous from the right, as are derivatives of elements in $V_h(\Omega)$. In order to incorporate Dirichlet boundary conditions we define the space

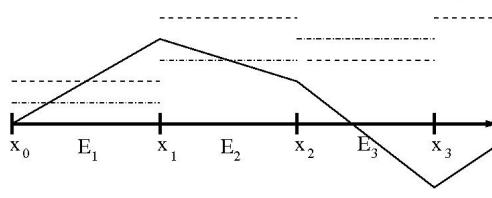$$V_h^0(\Omega) = \{ v \in V_h(\Omega); v|_{\partial\Omega} = 0 \} .$$

Figure 1: A solution of the scheme at a fixed time. The specific volume τ^h and temperature θ^h are piecewise constant while the velocity u^h is piecewise linear. Note that the nodes are numbered from 0 while the elements are numbered from 1.

For $Q_h(\Omega)$ we define the projection operator $\Pi_h^Q : L^p(\Omega) \to Q_h(\Omega)$, $p \in [1, \infty]$ by

$$\int_E [\Pi_h^Q \phi](x) \; dx = \int_E \phi(x) \; dx, \quad \forall E \in \mathbb{E}_h \,. \tag{3.1}$$

For $V_h(\Omega)$ we define the projection operator $\Pi_h^V : W^{1,2}(\Omega) \to V_h(\Omega)$ by

$$[\Pi_h^V \phi](x_i) = \phi(x_i), \quad i = 0, \dots, N \,,$$

such that

$$\left(\Pi_h^V \phi\right)_x = \Pi_h^Q(\phi_x) \,. \tag{3.2}$$

For any $q \in Q_h(\Omega)$ we set $[\![q]\!]_i := q|_{E_{i+1}} - q|_{E_i}$, where E_i denotes the ith element of $\mathbb{E}_h$.

Definition 3.1 (Semi-discrete finite element scheme) *Let the constitutive relations (1.5) and (2.6) hold and fix $h > 0$ with $N := \frac{|\Omega|}{h} \in \mathbb{N}$. For initial data (τ_0, u_0, θ_0) satisfying (2.2) we set*

$$(\tau_0^h, u_0^h, \theta_0^h) := (\Pi_h^Q [\tau_0], \Pi_h[u_0], \Pi_h^Q[\theta_0]), \tag{3.3}$$

where $\Pi_h : L^p(\Omega) \to V^h(\Omega)$ is the L^2-projection. For a given time $T > 0$ we determine functions

$$(\tau^h, u^h, \theta^h)(t) \in Q_h(\Omega) \times V_h^0(\Omega) \times V_h(\Omega), \qquad t \in (0, T),$$

such that

$$\int_\Omega \tau_t^h \phi^h \; dx = \int_\Omega u_x^h \phi^h \; dx \qquad \forall \phi^h \in Q_h(\Omega) \tag{3.4}$$

$$\int_\Omega u_t^h v^h \; dx = -\int_\Omega \left[\frac{\mu(\theta^h) u_x^h}{\tau^h} - p(\theta^h, \tau^h)\right] v_x^h \; dx \qquad \forall v^h \in V_h^0(\Omega) \tag{3.5}$$

$$\int_\Omega \theta_t^h \psi^h \; dx + \frac{1}{h} \sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i [\![\psi^h]\!]_i$$

$$= \int_\Omega \left[\frac{\mu(\theta^h)|u_x^h|^2}{\tau^h} - p(\theta^h, \tau^h) u_x^h\right] \psi^h \; dx \qquad \forall \psi^h \in Q_h(\Omega), \tag{3.6}$$

where $G_i(\tau^h) = 2\left(\tau^h|_{E_i} + \tau^h|_{E_{i+1}}\right)^{-1}$ and $L(z)$ is given by (2.7).

3.2 Basic estimates and well-posedness of the scheme

To show that the scheme is well-defined we recast (3.4)-(3.6) as an ODE on the (finite dimensional) finite element space $Q_h(\Omega) \times V_h^0(\Omega) \times Q_h(\Omega)$. Assuming for

6

now that a solution (τ^h, u^h, θ^h) exists, we show mass and energy conservation, and entropy balance, for the scheme. With $\phi^h = 1$ equation (3.4) gives

$$\int_\Omega \tau^h(x,t)\ dx \equiv \int_\Omega \tau_0^h(x)\ dx\,. \tag{3.7}$$

Next, (3.5) with $v^h = u^h$ and (3.6) with $\psi^h = 1$ give conservation of energy:

$$\mathcal{E}(u^h, \theta^h) := \int_\Omega \left(\tfrac{1}{2}|u^h|^2 + \theta^h\right)(x,t)\ dx \equiv \mathcal{E}(u_0^h, \theta_0^h)\,. \tag{3.8}$$

Finally, (3.6) with $\psi^h = \frac{1}{\theta^h}$ and (3.4) with $\phi^h := \frac{K}{\tau^h}$ give

$$\frac{d}{dt}\int_\Omega \log\theta^h + K\log\tau^h\ dx = \int_\Omega \frac{\mu|u_x^h|^2}{\tau^h\theta^h}\ dx - \frac{1}{h}\sum_{i=1}^{N-1} G_i(\tau^h)[\![L'(\theta^h)]\!]_i \left[\!\!\left[\frac{1}{\theta^h}\right]\!\!\right]_i$$

$$= \int_\Omega \frac{\mu|u_x^h|^2}{\tau^h\theta^h}\ dx + \frac{1}{h}\sum_{i=1}^{N-1} \frac{G_i(\tau^h)\kappa(\theta_{i\dagger}^h)}{(\theta_{i*}^h)^2}[\![\theta^h]\!]_i^2 \tag{3.9}$$

where, for each $i = 1,\ldots, N-1$, $\theta_{i\dagger}^h$, θ_{i*}^h lie between $\theta^h|_{E_i}$ and $\theta^h|_{E_{i+1}}$ (obtained from the mean value theorem applied to $L'(\theta)$ and $\frac{1}{\theta}$, respectively). Thus the discrete entropy

$$\mathcal{S}(\tau^h, u^h, \theta^h) := \tfrac{1}{2}|u^h|^2 + \theta^h + K\tau^h - \log(\theta^h) - K\log(\tau^h)$$

satisfies

$$\frac{d}{dt}\int_\Omega \mathcal{S}(\tau^h, u^h, \theta^h)\ dx + \int_\Omega \frac{\mu|u_x^h|^2}{\tau^h\theta^h}\ dx + \frac{1}{h}\sum_{i=1}^{N-1} \frac{G_i(\tau^h)\kappa(\theta_{i\dagger}^h)}{(\theta_{i*}^h)^2}[\![\theta^h]\!]_i^2 = 0\,. \tag{3.10}$$

Lemma 3.1 (Well-posedness of approximation scheme) *Under the same assumptions as in Definition 3.1 there exists a unique solution (τ^h, u^h, θ^h) of the approximation scheme (3.4)-(3.6). The solution is defined for all times $t > 0$ and there is a constant C_0, depending on h and (τ_0, u_0, θ_0), but independent of time, such that*

$$C_0^{-1} \le \tau^h(x,t),\ \theta^h(x,t) \le C_0 \qquad \forall x \in \Omega\,,\ \forall t \ge 0\,.$$

Proof: We define the finite element space $\boldsymbol{S}_h(\Omega) := Q_h(\Omega) \times V_h^0(\Omega) \times Q_h(\Omega)$ of (τ^h, u^h, θ^h) triples and equip it with the standard $\boldsymbol{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ norm. Let $\boldsymbol{S}_h^+(\Omega)$ denote the subset $\boldsymbol{S}_h(\Omega) \cap \{\tau^h,\ \theta^h > 0\}$. For each $\boldsymbol{z} = (z_1, z_2, z_3) \in \boldsymbol{S}_h^+(\Omega)$ we define $\boldsymbol{F}(\boldsymbol{z})$ to be the unique element in $\boldsymbol{S}_h(\Omega)$ satisfying

$$\langle \boldsymbol{F}(\boldsymbol{z}), \boldsymbol{q}\rangle = \int_\Omega (z_2)_x q_1\ dx + \int_\Omega \left[\frac{\mu(z_3)(z_2)_x}{z_1} - p(z_3, z_1)\right]\left[(z_2)_x q_3 - (q_2)_x\right]\ dx$$

$$- \frac{1}{h}\sum_{i=1}^{N-1} G_i(z_1)[\![L'(z_3)]\!]_i[\![q_3]\!]_i, \tag{3.11}$$

for all $\boldsymbol{q} = (q_1, q_2, q_3) \in \boldsymbol{S}_h(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\boldsymbol{L}^2(\Omega)$. Linearity with respect to $\boldsymbol{q}$ in (3.11) implies that the approximation scheme (3.4)-(3.6) corresponds to the ODE

$$\boldsymbol{z}_t = \boldsymbol{F}(\boldsymbol{z}), \quad \boldsymbol{z}(0) = \boldsymbol{z}_0^h = (\tau_0^h, u_0^h, \theta_0^h). \tag{3.12}$$

For the given initial data $\boldsymbol{z}_0^h$ we define the compact subset

$$\boldsymbol{K}(\boldsymbol{z}_0^h):=\Big\{\boldsymbol{z} \in \boldsymbol{S}_h^+(\Omega); \int_\Omega z_1 \; dx \le \int_\Omega \tau_0^h \; dx, \int_\Omega \mathcal{E}(z_2, z_3) \; dx \le \int_\Omega \mathcal{E}(u_0^h, \theta_0^h) \; dx,$$
$$\int_\Omega \log\big(z_3 z_1^K\big) \; dx \ge \int_\Omega \log\big(\theta_0^h (\tau_0^h)^K\big) \; dx\Big\}.$$

It follows from the estimates (3.7)-(3.10) that any solution $\boldsymbol{z}(t)$ of the initial value problem (3.12) belongs to $\boldsymbol{K}(\boldsymbol{z}_0^h)$ for all times it is defined. In turn this gives a time independent (but h-dependent) upper bound for $\|\boldsymbol{z}(t)\|_{\boldsymbol{L}^\infty(\Omega)}$, as well as for $\|z_1^{-1}(t)\|_{L^\infty(\Omega)}$ and $\|z_3^{-1}(t)\|_{L^\infty(\Omega)}$. A direct calculation shows that, thanks to these bounds, $\boldsymbol{F}$ is uniformly Lipschitz continuous on $\boldsymbol{K}(\boldsymbol{z}_0^h)$: there is a constant L (depending on h and $\boldsymbol{z}_0^h$) such that

$$\|\boldsymbol{F}(\boldsymbol{z}) - \boldsymbol{F}(\boldsymbol{y})\|_{\boldsymbol{L}^2(\Omega)} \le L \|\boldsymbol{z} - \boldsymbol{y}\|_{\boldsymbol{L}^2(\Omega)}, \quad \forall \boldsymbol{z}, \boldsymbol{y} \in \boldsymbol{K}(\boldsymbol{z}_0^h).$$

Standard ODE theory now gives existence and uniqueness of the solution $\boldsymbol{z}(t)$ to (3.12) for all times $t \ge 0$. I.e., $(\tau^h, u^h, \theta^h)(t) = (z_1(t), z_2(t), z_3(t))$ is the unique solution to the approximation scheme (3.4)-(3.6) with initial data $\boldsymbol{z}_0^h$. $\qquad\square$

4 Convergence of the approximation scheme

In this section we formulate certain (strong) apriori estimates on solutions of the scheme (3.4)-(3.6), and we show that these are sufficient to conclude convergence to a weak solution of the system (1.1), (1.2), (2.1). As noted in the introduction, this part of the analysis works for more general powers α and β (see (2.6)) than those assumed in the statement of Theorem 2.1. On the other hand we recall that we restrict ourselves to ideal polytropic gases.

Notation 4.1 *Given a set of elements $\{v^h\}$ in a normed space $(X, \|\cdot\|)$, we write "$v^h \in_b X$" to mean that $\|v^h\|$ is bounded independently of h. Weak convergence is denoted by $\rightharpoonup$ and weak-* convergence is denoted $\overset{\star}{\rightharpoonup}$. An over bar denotes weak L^1-limit, see e.g. (4.15). Subsequences are not relabeled. A zero subscript denotes evaluation at time $t = 0$.*

For later reference we recall the following general results:

Lemma 4.1 *Let $\{f^h\}$, $\{g^h\}$ be sequences of functions on a measurable subset $\mathcal{O}$ of $\Omega \times (0, T)$. Assume that $f^h \in_b L^\infty(\mathcal{O})$ with $f^h \to f$ a.e. in $\mathcal{O}$, and $g^h \in_b L^1(\mathcal{O})$ with $g^h \rightharpoonup g$ in $L^1(\mathcal{O})$. Then $f^h g^h \rightharpoonup fg$ in $L^1(\mathcal{O})$.*

Lemma 4.2 *([9, Theorem 2.11]) Let $O \subset \mathbb{R}^M$, $M \ge 1$, be bounded and open and let $g \colon \mathbb{R} \to \mathbb{R}$ be continuous and convex. Let $\{v_n\}_{n \ge 1}$ be a sequence in $L^1(O)$ such that $v_n \rightharpoonup v$ in $L^1(O)$, $g(v_n) \in L^1(O)$ for each n, and $g(v_n) \rightharpoonup \overline{g(v)}$ in $L^1(O)$. Then $g(v) \le \overline{g(v)}$ a.e. on O and $g(v) \in L^1(O)$. If, in addition, g is strictly convex on an open interval $(a, b) \subset \mathbb{R}$ and $g(v) = \overline{g(v)}$ a.e. on O, then, passing to a subsequence if necessary, $v_n(x) \to v(x)$ for a.e. $x \in \{y \in O \mid v(y) \in (a, b)\}$.*

Lemma 4.3 *([23, Corollary 4]) Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \le p < \infty$, $\{v \colon v \in L^p(0, T; X), v_t \in L^1(0, T; Y)\}$ is compactly embedded in $L^p(0, T; B)$. In the case $p = \infty$, for any $r > 1$, $\{v \colon v \in L^\infty(0, T; X), v_t \in L^r(0, T; Y)\}$ is compactly embedded in $C(0, T; B)$.*

The main result in this section is:

Theorem 4.1 *Assume the constitutive relations (1.5) and (2.6) with $0 \leq \beta < 2$. Let $\{(\tau^h, u^h, \theta^h)\}$ be a sequence of functions constructed according to Definition 3.1 for $t \in [0, T]$, and assume there exists $C > 0$, independent of h, such that*

$$[\mathbf{C1}]: \quad \left\{ \begin{array}{ccc} C^{-1} & \leq & \tau^h(x,t), \ \theta^h(x,t) \\ \mu(\theta^h(x,t)) & \leq & C \end{array} \right\} \quad \forall (x,t) \in \Omega \times [0,T)$$

$$[\mathbf{C2}]: \quad \sup_{t \in (0,T)} \|\theta^h(t)\|_{L^2(\Omega)} + \sum_{i=1}^{N-1} \int_0^T \frac{[\![\theta^h(t)]\!]_i^2}{h} \, dt \leq C$$

$$[\mathbf{C3}]: \quad u^h \in_b L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)).$$

Then, passing to a subsequence if necessary, $(\tau^h, u^h, \theta^h) \to (\tau, u, \theta)$ a.e in $\Omega \times (0,T)$, where (τ, u, θ) is a weak solution to the compressible Navier–Stokes system (1.1) - (2.1) in the sense of Definition 2.1.

This theorem will be a consequence of Lemmas 4.4 - 4.7.

Remark 4.1 *Clearly, $[\mathbf{C1}]$ is satisfied for $\alpha = 0$ while it is equivalent to a uniform upper bound on θ^h when $\alpha > 0$.*

By $[\mathbf{C1}] - [\mathbf{C3}]$ there exist functions τ, u, θ such that

$$u^h \overset{\star}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)) \tag{4.1}$$

$$\tau^h \rightharpoonup \tau \quad \text{in} \quad W^{1,2}(0,T;L^2(\Omega)) \tag{4.2}$$

$$\theta^h \overset{\star}{\rightharpoonup} \theta \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \tag{4.3}$$

if necessary by passing to a subsequence. Here (4.1) follows from $[\mathbf{C3}]$, (4.2) follows from $\tau_t^h = u_x^h$ and $[\mathbf{C3}]$, and (4.3) follows from $[\mathbf{C2}]$.

We will show that the triple (τ, u, θ) is a weak solution of the Navier-Stokes system. In Section 4.1 we prove that the velocity and temperature converge strongly by utilizing Lemma 4.3. Then a convexity argument is used repeatedly in Section 4.2 to show that the specific volume converges strongly. At this point it follows that (τ, u, θ) is a weak solution of (1.1) and (1.2). Exploiting this and the fact that the pressure is an L^2 function, we find that $\{u_x^h\}$ converges strongly. It then follows that the limits satisfy the temperature equation as well, and Theorem 4.1 follows.

4.1 Strong convergence of u^h and θ^h

Lemma 4.4 *Let $\{\tau^h, u^h, \theta^h\}$ be as in Theorem 4.1. Then*

$$u_t^h \in_b L^2(0,T;W^{-1,2}(\Omega)), \tag{4.4}$$

and consequently, if necessary passing to a subsequence,

$$u^h \to u \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{4.5}$$

Proof: Fix $v \in L^2(0,T;W_0^{1,2}(\Omega))$ and define functions $v^h \in V_h^0(\Omega)$ by requiring

$$\int_0^T \int_\Omega v^h w^h \, dxdt = \int_0^T \int_\Omega v w^h \, dxdt \quad \forall w^h \in V_h^0(\Omega),$$

i.e., v^h is the L^2-projection of v onto $V_h^0(\Omega)$. Then, by [2], there exists a constant $C > 0$, independent of h, such that

$$\|v^h\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C\|v\|_{L^2(0,T;W^{1,2}(\Omega))}.$$

9

Using v^h as test function in the momentum scheme (3.5) gives

$$\left|\int_0^T \int_\Omega u_t^h v \; dxdt\right| = \left|\int_0^T \int_\Omega u_t^h v^h \; dxdt\right|$$

$$\leq \|v^h\|_{L^2(0,T;W^{1,2}(\Omega))} \left[\left\|\frac{\mu(\theta^h)}{\tau^h} u_x^h\right\|_{L^2(0,T;L^2(\Omega))} + \|p(\theta^h, \tau^h)\|_{L^2(0,T;L^2(\Omega))}\right]$$

$$\leq C\|v\|_{L^2(0,T;W^{1,2}(\Omega))},$$

where we have used [C1] - [C3]. As v is arbitrary, we conclude (4.4).

In the current situation where both [C3] and (4.4) holds, Lemma 4.3, with $X = W_0^{1,2}(\Omega)$, $B = L^2(\Omega)$, and $Y = W^{-1,2}(\Omega)$, yields (4.5). $\qquad\square$

To analyze the convergence of θ^h we define a dual mesh $\mathbb{E}_h^\perp$ with vertices

$$x_i^\perp = h\left(i - \tfrac{1}{2}\right), \; i = 1, \ldots, N-1, \quad x_0^\perp = 0, \quad x_N^\perp = L.$$

We define finite element spaces $V_h^\perp(\Omega)$, $Q_h^\perp(\Omega)$ and corresponding projections $\Pi_h^{V^\perp}$ and $\Pi_h^{Q^\perp}$ exactly as we did for $\mathbb{E}_h$. To shorten notation we write

$$\widetilde{\phi^h} = \Pi_h^{V^\perp}[\phi]. \tag{4.6}$$

Lemma 4.5 *Let* $\left\{(\tau^h, u^h, \theta^h)\right\}$ *be as in Theorem 4.1. Then,*

$$\widetilde{\theta^h} \in_b L^2(0,T; W^{1,2}(\Omega)), \qquad \theta_t^h \in_b L^1(0,T; W^{-1,1}(\Omega)), \tag{4.7}$$

and $\theta^h \to \theta$ *in* $L^p(0,T; L^p(\Omega))$, $\forall p \in [1, 4)$, *if necessary passing to a subsequence.*

Proof: The dual mesh is defined such that:

$$h^{-1}[\![\theta^h]\!]_i = \widetilde{\theta^h}_x, \qquad x \in (x_i^\perp, x_{i+1}^\perp), \quad i = 1, \ldots, N-1.$$

and by [C2] there is a constant $C > 0$, independent of h, such that

$$\|\widetilde{\theta^h}_x\|_{L^2(0,T;L^2(\Omega))}^2 = \int_0^T \sum_{i=1}^{N-1} \frac{[\![\theta^h]\!]_i^2}{h} \; dt \leq C. \tag{4.8}$$

Since $\|\widetilde{\theta^h}(t) - \theta^h(t)\|_{L^2(\Omega)} \leq Ch\|\widetilde{\theta^h_x}(t)\|_{L^2(\Omega)}$, we thus have

$$\|\widetilde{\theta^h} - \theta^h\|_{L^2(0,T;L^2(\Omega))} \leq Ch\|\widetilde{\theta^h}_x\|_{L^2(0,T;L^2(\Omega))} \leq Ch,$$

and [C2] gives

$$\|\widetilde{\theta^h}\|_{L^2(0,T;L^2(\Omega))} \leq \|\widetilde{\theta^h} - \theta^h\|_{L^2(0,T;L^2(\Omega))} + \|\theta^h\|_{L^2(0,T;L^2(\Omega))} \leq C(1 + h),$$

which proves (4.7)$_1$. For later reference we note that, again by (4.8), $\widetilde{\theta^h} \in_b L^2(0,T; L^\infty(\Omega))$, whence the same holds for θ^h. Together with [C2] and the Cauchy-Schwarz inequality this gives

$$\theta^h \in_b L^4(0,T; L^4(\Omega)), \tag{4.9}$$

which in turn shows that $\kappa(\theta^h) \in_b L^2(0,T; L^2(\Omega))$, since $\beta < 2$.

For (4.7)$_2$ we fix $\psi \in W^{1,\infty}(\Omega)$ and define the discrete effective viscous flux by $\mathcal{F}^h := \frac{\mu(\theta^h)}{\tau^h} u_x^h - p(\theta^h, \tau^h)$. By [C1] - [C3], and the Cauchy-Schwarz

inequality we have $\mathcal{F}^h \in_{\mathrm{b}} L^2(0, T; L^2(\Omega))$. Using $\Pi_h^Q \psi$ as test function in the temperature approximation (3.6), we deduce

$$\left| \int_\Omega \theta_t^h \psi \; dx \right| = \left| \int_\Omega \mathcal{F}^h u_x^h \psi \; dx - \frac{1}{h} \sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i [\![\Pi_h^Q \psi]\!]_i \right|$$

$$\leq \|\psi\|_{L^\infty(\Omega)} \|\mathcal{F}^h u_x^h\|_{L^1(\Omega)} + \|\psi_x\|_{L^\infty(\Omega)} \sum_{i=1}^{N-1} G_i(\tau^h) \left| [\![L'(\theta^h)]\!]_i \right| \qquad (4.10)$$

$$\leq \|\psi\|_{W^{1,\infty}(\Omega)} \left[\|\mathcal{F}^h\|_{L^2(\Omega)} \|u_x^h\|_{L^2(\Omega)} + C \|\kappa(\theta^h)\|_{L^2(\Omega)} \left[\sum_{i=1}^{N-1} \frac{1}{h} [\![\theta^h]\!]_i^2 \right]^{\frac{1}{2}} \right],$$

where we have used [**C1**] and that $L''(z) = \kappa(z)$. Integrating (4.10) and applying [**C2**]-[**C3**], together with the bounds on $\mathcal{F}^h$ and $\kappa(\theta^h)$, then gives $(4.7)_2$.

Next we establish compactness of $\{\theta^h\}$ with respect to the spatial variable. Let $\xi \in (-L, L)$ be arbitrary and define the domain $\Omega_\xi = (|\xi|, L - |\xi|)$. If $h \leq |\xi|$, fix any $x \in \Omega_\xi$, let E_i be the element containing x, and let E_j be the element containing $x - \xi$. Then

$$|\theta^h(x, t) - \theta^h(x - \xi, t)|^2 = |\theta^h(t)|_{E_i} - \theta^h(t)|_{E_j}|^2$$

$$= \left| \pm \sum_{k=i}^{j-1} h^{\frac{1}{2}} h^{-\frac{1}{2}} [\![\theta^h]\!]_k \right|^2 \leq \left[\sum_{k=i}^{j-1} h \right] \left[\sum_{k=1}^{N-1} \frac{[\![\theta^h]\!]_k^2}{h} \right] \leq C(|\xi| + h) \left[\sum_{k=1}^{N-1} \frac{[\![\theta^h]\!]_k^2}{h} \right].$$

Integrating over $\Omega_\xi \times (0, T)$, and using $h \leq |\xi|$ together with [**C2**], we obtain

$$\|\theta^h(x, t) - \theta^h(x - \xi, t)\|_{L^2(0, T; L^2(\Omega_\xi))} \leq C |\xi|^{\frac{1}{2}}. \qquad (4.11)$$

On the other hand, if $|\xi| \leq h$, then for each $0 \leq t \leq T$ we have (for $h < 1$)

$$\int_0^T \int_{\Omega_\xi} |\theta^h(x, t) - \theta^h(x - \xi, t)|^2 dx dt = \int_0^T |\xi| \sum_{i=1}^{N-1} [\![\theta^h]\!]_i^2 \, dt \leq |\xi| \int_0^T \sum_{i=1}^{N-1} \frac{[\![\theta^h]\!]_i^2}{h} \, dt,$$

whence [**C2**] shows that (4.11) holds also for $|\xi| \leq h$. From the translation estimate (4.11), it follows that $\theta^h \in_{\mathrm{b}} L^2(0, T; W^{\frac{1}{2} - \epsilon, 2}(\Omega))$, see [1].

Consequently, Lemma 4.3, with $X = W^{\frac{1}{2} - \epsilon, 2}(\Omega)$, $\epsilon > 0$, $B = L^2(\Omega)$, and $Y = W^{-1,1}(\Omega)$, yields

$$\theta^h \to \theta \qquad \text{in } L^2(0, T; L^2(\Omega)) . \qquad (4.12)$$

Finally, (4.9) and (4.12) concludes the proof. $\qquad \square$

We observe that (4.12) and [**C1**] yields:

$$\mu(\theta^h) \to \mu(\theta) \qquad \text{in} \quad L^p(\Omega \times (0, T)) \qquad \forall \, 1 \leq p < \infty. \qquad (4.13)$$

4.2 Strong convergence of τ^h

We first note that $\tau_t^h = u_x^h$ a.e. by (3.4), whence (4.1) and (4.2) show that τ, u satisfy (2.3). Passing to the limit a.e. in the other equations requires strong convergence of τ^h. We begin by observing that, due to [**C1**], we may divide by τ in (2.3) to obtain the "renormalized" equation (used below in (4.19))

$$(\log \tau)_t = \frac{u_x}{\tau} \qquad \text{a.e. in } \Omega \times (0, T). \qquad (4.14)$$

We proceed to adapt a simplified version of the convexity arguments in [17], [10] to the Lagrangian setting and obtain the required strong convergence.

11

Lemma 4.6 *Let $\{\tau^h, u^h, \theta^h\}$ be as in Theorem 4.1. Then,*

$$\tau^h \to \tau \qquad a.e. \ in \ \Omega \times (0,T).$$

Proof: Step 1: Recalling the notation for weak L^1-limits we claim that

$$\overline{\left(\frac{u_x}{\tau}\right)} - \frac{u_x}{\tau} \geq 0 \qquad \text{a.e. in } \Omega \times (0,T). \tag{4.15}$$

For the proof we fix $\psi \in C_c^\infty(\Omega \times (0,T)) \cap \{\psi \geq 0\}$ and set

$$v^h(x,t) := \int_0^x \left[\psi \frac{\tau - \tau^h}{\tau}\right](y,t)\,dy - \frac{x}{|\Omega|}\int_\Omega \left[\psi \frac{\tau - \tau^h}{\tau}\right](y,t)\,dy,$$

where the integrand is differentiable in time since

$$\left[\frac{\tau - \tau^h}{\tau}\right]_t = \frac{\tau^h u_x - \tau u_x^h}{\tau^2} \in_{\mathrm{b}} L^2(0,T;L^1(\Omega)). \tag{4.16}$$

Hence, $v_t^h \in_{\mathrm{b}} L^2(0,T;L^\infty(\Omega))$, and by (4.1) and (4.2) we have

$$v_t^h \overset{\star}{\rightharpoonup} 0 \qquad \text{in } L^2(0,T;L^\infty(\Omega)). \tag{4.17}$$

We then use $\Pi_h^V[v^h]$ as test function (3.5) and integrate in time to obtain

$$\int_0^T \int_\Omega \left[\frac{\mu(\theta^h)u_x^h}{\tau^h} - \frac{\mu(\theta^h)u_x^h}{\tau}\right]\psi\,dxdt - \int_0^T\int_\Omega \psi\left[p(\theta^h,\tau^h) - p(\theta^h,\tau)\right]dxdt$$

$$= \int_0^T\left[\left[\int_\Omega \frac{\mu(\theta^h)u_x^h}{\tau^h} - p(\theta^h,\tau^h)\,dx\right]\left[\frac{1}{|\Omega|}\int_\Omega \psi\frac{\tau-\tau^h}{\tau}\,dy\right]\right]dt$$

$$+ \int_0^T\int_\Omega u^h v_t^h\,dxdt + \int_0^T\int_\Omega u^h\left[\Pi_h^V[v_t^h] - v_t^h\right]dxdt =: \sum_{i=1}^3 I_i, \tag{4.18}$$

where we have used (3.1) and (3.2). We observe that $I_1, I_2 \to 0$ as $h \to 0$, due to (4.17) and Lemma 4.4. The I_3 term satisfies

$$|I_3| \leq \|u^h\|_{L^2(0,T;L^\infty(\Omega))}\|\Pi_h^V[v_t^h] - v_t^h\|_{L^2(0,T;L^1(\Omega))},$$

where the first factor is bounded by [**C3**]. A standard interpolation estimate [3], together with (4.16), gives

$$\|\Pi_h^V[v_t^h] - v_t^h\|_{L^2(0,T;L^1(\Omega))} \leq Ch\|v_{tx}^h\|_{L^2(0,T;L^1(\Omega))} \leq Ch.$$

Recalling (4.13) and Lemma 4.5, and sending $h \to 0$ in (4.18) we get

$$\lim_{h\to0}\int_0^T\int_\Omega \mu(\theta)\left[\frac{u_x^h}{\tau^h} - \frac{u_x}{\tau}\right]\psi\,dxdt = \lim_{h\to0}\int_0^T\int_\Omega K\theta\left[\frac{1}{\tau^h} - \frac{1}{\tau}\right]\psi\,dxdt \geq 0,$$

where the last inequality follows from Lemma 4.2 and the convexity of $z \mapsto \frac{1}{z}$. We thus have

$$\mu(\theta)\left[\overline{\left[\frac{u_x}{\tau}\right]} - \frac{u_x}{\tau}\right] \geq 0 \qquad \text{a.e in } \Omega \times (0,T),$$

and the claim (4.15) follows from the lower bound [**C1**] on $\mu(\theta)$.

Step 2: By (4.14) and (4.15) it follows that

$$\left[\overline{\log\tau} - \log\tau\right]_t = \overline{\left[\frac{u_x}{\tau}\right]} - \frac{u_x}{\tau} \geq 0 \qquad \text{a.e. in } (0,T) \times \Omega. \tag{4.19}$$

As $\log\tau_0^h \to \log\tau_0$ a.e. in Ω we conclude that

$$\overline{\log\tau}(x,t) - \log\tau(x,t) \geq 0 \qquad \text{for a.e } (x,t) \in \Omega \times (0,T).$$

On the other hand, Lemma 4.2 shows that $\overline{\log\tau}(x,t) - \log\tau(x,t) \leq 0$ a.e. Thus, $\overline{\log\tau} = \log\tau$ a.e. in $\Omega \times (0,T)$, and the conclusion follows by an application of the last part of Lemma 4.2. $\qquad\square$

4.3 Concluding the proof of Theorem 4.1

In view of the two previous lemmas and [**C1**],

$$p(\theta^h, \tau^h) \to p(\theta, \tau) \qquad \text{in } L^2(0, T; L^2(\Omega)),$$

$$\frac{\mu(\theta^h)}{\tau^h} \to \frac{\mu(\theta)}{\tau} \qquad \text{in } L^p(\Omega \times (0, T)), \forall 1 \le p < \infty. \tag{4.20}$$

Given $\phi \in C_c^\infty(\Omega \times [0, T))$ we use (3.5) with $v^h = \Pi_h^V[\phi]$ and integrate in time:

$$\int_0^T \int_\Omega u^h \Pi_h^V[\phi_t] \, dxdt + \int_\Omega u_0^h \Pi_h^V[\phi_0] \, dx = \int_0^T \int_\Omega \mathcal{F}^h \Pi_h^Q[\phi_x] \, dxdt. \tag{4.21}$$

Due to the regularity of ϕ, standard interpolation estimates show, together with [**C0**], (4.1), and (4.20), that the limit $h \to 0$ in (4.21) results in (2.4). By now we have established the a.e. convergence $(\tau^h, u^h, \theta^h) \to (\tau, u, \theta)$, and that the latter triple satisfies (2.3) and (2.4). It remains to prove that (τ, u, θ) also satisfies (2.5). In order to pass to the limit in the temperature scheme, the following lemma is essential; its proof exploits the fact that (τ, u, θ) satisfies (2.4).

Lemma 4.7 *Let* $\left\{\tau^h, u^h, \theta^h\right\}$ *be as in Theorem 4.1. Then,*

$$\lim_{h \to 0} \left[\sup_{t \in (0, T)} \int_\Omega \frac{|u^h - u|^2}{2} \, dx + \int_0^T \int_\Omega \frac{\mu(\theta^h)}{\tau^h} |u_x^h - u_x|^2 \, dxdt \right] = 0.$$

Proof: Using u^h as test function in (3.5) and integrating in time we get

$$\int_\Omega \frac{|u^h(t)|^2}{2} \, dx = \int_\Omega \frac{|u_0^h|^2}{2} \, dx - \int_0^t \int_\Omega \mu(\theta^h) \frac{|u_x^h|^2}{\tau^h} - p(\theta^h, \tau^h) u_x^h \, dxdt. \tag{4.22}$$

Sending $h \to 0$ in (4.22), and using (4.20), gives

$$\lim_{h \to 0} \left[\frac{\|u^h(t)\|_2^2}{2} + \int_0^t \int_\Omega \mu(\theta^h) \frac{|u_x^h|^2}{\tau^h} \, dxdt \right] = \frac{\|u_0^h\|_2^2}{2} + \int_0^t \int_\Omega p(\theta, \tau) u_x \, dxdt.$$

On the other hand, by Lemma 4.4, $u_t \in_b L^2(0, T; W^{-1,2}(\Omega))$. This and the weak form of the momentum equation (2.4) yield the weak form

$$\int_0^t \langle u_t, v \rangle_{\langle W^{-1,2}, W_0^{1,2} \rangle} \, dt = -\int_0^t \int_\Omega \left[\frac{\mu(\theta) u_x}{\tau} - p(\theta, \tau) \right] v_x \, dxdt,$$

valid for all $v \in L^2(0, T; W_0^{1,2}(\Omega))$. Applying this with $v = u$ gives

$$\frac{\|u(t)\|_2^2}{2} + \int_0^t \int_\Omega \mu(\theta) \frac{|u_x|^2}{\tau} \, dxdt = \frac{\|u_0\|_2^2}{2} + \int_0^t \int_\Omega p(\theta, \tau) u_x \, dxdt,$$

whence

$$\lim_{h \to 0} \left[\frac{\|u(t)\|_2^2}{2} - \frac{\|u_0\|_2^2}{2} + \int_0^t \int_\Omega \mu(\theta^h) \frac{|u_x^h|^2}{\tau^h} - \mu(\theta) \frac{|u_x|^2}{\tau} \, dxdt \right] = 0 \quad \forall t \in (0, T).$$

Next, $u_x^h \rightharpoonup u_x$ in $L^2(0, T; L^2(\Omega))$ and $\frac{\mu(\theta^h)}{\tau^h} \to \frac{\mu(\theta)}{\tau}$ in $L^p(0, T; L^p(\Omega))$ for any $p < \infty$. Consequently, in view of Lemma 4.1,

$$\lim_{h \to 0} \int_0^T \int_\Omega \frac{\mu(\theta^h)}{\tau^h} u_x^h u_x \, dxdt = \int_0^T \int_\Omega \frac{\mu(\theta)}{\tau} u_x^2 \, dxdt,$$

such that

$$\lim_{h \to 0} \left[\sup_{t \in (0,T)} \int_\Omega \frac{|u^h - u|^2}{2} \, dx + \int_0^T \int_\Omega \frac{\mu(\theta^h)}{\tau^h} |u_x^h - u_x|^2 \, dxdt \right]$$

$$= \lim_{h \to 0} \left[\sup_{t \in (0,T)} \int_\Omega \frac{|u^h|^2 - |u|^2}{2} \, dx + \int_0^T \int_\Omega \mu(\theta^h) \frac{|u_x^h|^2}{\tau^h} - \mu(\theta) \frac{|u_x|^2}{\tau} \, dxdt \right] = 0 \,.$$

As the left hand side is non-negative this concludes the proof. □

Finally, fix any $\phi \in C_c^\infty(\Omega \times [0,T))$. A calculation using (4.6) shows that

$$\sum_{i=1}^{N-1} \frac{1}{h} G_i(\tau^h) [\![L'(\theta^h)]\!]_i [\![\Pi_h^Q [\phi]]\!]_i = \int_\Omega \frac{\kappa(\theta_{i\dagger}^h)}{\Pi_h^{Q\perp}[\tau^h]} \big(\widetilde{\theta^h}\big)_x \big(\widetilde{\Pi_h^Q [\phi]}\big)_x \, dx, \qquad (4.23)$$

where $\theta_{i\dagger}^h$ is as in (3.9). Now using $\Pi_h^Q [\phi]$ as test function in (3.6) we obtain

$$\int_0^T \int_\Omega \theta^h \phi_t \, dxdt - \int_\Omega \theta_0^h \phi_0 \, dx - \int_0^T \int_\Omega \frac{\kappa(\theta_{i\dagger}^h(x))}{\tau^h} \big(\widetilde{\theta^h}\big)_x \big(\widetilde{\Pi_h^Q [\phi]}\big)_x \, dxdt$$

$$= \int_0^T \int_\Omega \left[p(\theta^h, \tau^h) u_x^h - \frac{\mu(\theta^h)|u_x|^2}{\tau^h} - \frac{\mu(\theta^h) \big[|u_x^h|^2 - |u_x|^2\big]}{\tau^h} \right] \phi \, dxdt. \qquad (4.24)$$

On each interval $(x_i^\perp, x_{i+1}^\perp)$, $\kappa(\theta_{i\dagger}^h)$ is a linear combination of $\kappa(\theta_i^h)$ and $\kappa(\theta_{i+1}^h)$. Hence, by Lemma 4.5 and since $\beta < 2$, $\kappa(\theta_{i\dagger}^h) \to \kappa(\theta)$ in $L^2(0,T;L^2(\Omega))$. Thus $\kappa(\theta_{i\dagger}^h(x))(\widetilde{\theta^h})_x \rightharpoonup \kappa(\theta)\theta_x$ in $L^1(0,T;L^1(\Omega))$. Also, as $\frac{1}{\tau^h} \to \frac{1}{\tau}$ in $L^p(0,T;L^p(\Omega))$ $\forall p \in [1,\infty)$, Lemma 4.1 gives

$$\frac{\kappa(\theta_{i\dagger}^h)}{\tau^h} \big(\widetilde{\theta^h}\big)_x \rightharpoonup \frac{\kappa(\theta)}{\tau} \theta_x \qquad \text{in } L^1(0,T;L^1(\Omega)) \,.$$

Letting $h \to 0$ in (4.24) and using Lemma 4.7 and (4.20), results in (2.5). This concludes the proof of Theorem 4.1.

5   Energy bounds and proof of Theorem 2.1

We now restrict to constitutive relations as described in Theorem 2.1, for which the pointwise estimates (2.10) are established in the next section. Taking these for granted we now prove the integral bounds (2.11). As demonstrated below these suffice, via Theorem 4.1, to establish Theorem 2.1. For reference we note:

[**A1**]:   We have $\alpha = 0$ and $\beta \in [0, \frac{3}{2})$ such that $\mathcal{F}^h$ and L are given by

$$\mathcal{F}^h = \frac{\bar{\mu}}{\tau^h} u_x^h - p(\theta^h, \tau^h) \qquad \text{and} \qquad L(\theta) = \frac{\bar{\kappa} \, \theta^{\beta+2}}{(\beta+1)(\beta+2)} \,;$$

[**A2**]: (τ^h, u^h, θ^h), $h > 0$, are solutions of the scheme (3.4)-(3.6) with initial data $(\tau_0^h, u_0^h, \theta_0^h)$ given by (3.3) with (τ_0, u_0, θ_0) as in Theorem 2.1.

For a fixed $T \in (0, \infty)$ we let C, $\tilde{C}$, etc. be numbers that depend on T, system parameters $(\Omega, \bar{\mu}, K)$, and the initial data, but that are independent of h.

Lemma 5.1 (Pointwise estimates) *Assume* [**A1**] *and* [**A2**]*. Then there exists a number $C > 0$ which is independent of h and such that:*

$$C^{-1} \le \tau^h(x,t) \le C, \quad \forall (x,t) \in \Omega \times (0,T), \qquad (5.1)$$

$$C^{-1} \leq \theta^h(x,t), \quad \forall (x,t) \in \Omega \times (0,T), \tag{5.2}$$

$$\int_0^T \|\theta^h(t)\|_{L^\infty(\Omega)} \, dt \leq C. \tag{5.3}$$

The proof of (5.1)-(5.3) is essentially the same as in [16]. Minor adjustments are required to treat the particular scheme (3.4)-(3.6) and to incorporate θ-dependence in the heat conductivity κ. For completeness we include the proof in Section 6. We note that these pointwise bounds do not seem to generalize in any simple way to the case of θ-dependent viscosities.

We proceed to state the discrete analogue of (2.11) (Lemma 5.2 below), and then show how this is used together with Lemma 5.1 to prove Theorem 2.1. The remaining parts of this section detail the proof of Lemma 5.2. A few technical lemmas are collected at the end of the section.

Lemma 5.2 (Energy estimates) *Define*

$$\mathcal{A}^h(t) := \sup_{s \in (0,t)} \int_\Omega |u_x^h|^2 \, dx + \int_0^t \int_\Omega |u_t^h|^2 \, dxds,$$

$$\mathcal{B}^h(t) := \sup_{s \in (0,t)} \int_\Omega L(\theta^h) \, dx + \frac{1}{h} \sum_{i=1}^{N-1} \int_0^t G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 \, ds,$$

$$\mathcal{D}^h(t) := \sup_{s \in (0,t)} \frac{1}{2h} \sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 + \int_0^t \int_\Omega \kappa(\theta^h) |\theta_t^h|^2 \, dxds.$$

Then there is a number $C > 0$, independent of h, such that

$$\mathcal{A}^h(t) + \mathcal{B}^h(t) + \mathcal{D}^h(t) \leq C \qquad \forall t \in (0,T). \tag{5.4}$$

5.1 Proof of Theorem 2.1

We now take Lemmas 5.1 and 5.2 for granted, and we verify that these are sufficient to verify the conditions in Theorem 4.1 (with $\alpha = 0$). First, our assumptions on the initial data in Theorem 2.1 are stronger than the corresponding conditions in Theorem 4.1. Next, [**C1**] is an immediate consequence of Lemma 5.1, [**C2**] follows from the bound on $\mathcal{B}^h$ together with the pointwise estimates of Lemma 5.1, and [**C3**] follows from the bound on $\mathcal{A}^h$. We can thus apply Theorem 4.1 and conclude the existence and convergence parts of Theorem 2.1. At this point, Lemma 5.1 yields (2.10).

It remains to verify (2.11), and we consider the first term in the $\mathcal{D}(t)$-functional in detail. By the $\mathcal{D}^h(t)$-bound in (5.4), we have as in (4.23) that

$$\int_\Omega \frac{|\kappa(\theta_{i\dagger}^h)|^2}{\Pi_h^{Q^\perp}[\tau^h]} |\widetilde{\theta^h}_x|^2 \, dx = \frac{1}{h} \sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 \leq C \qquad \forall t \in (0,T). \tag{5.5}$$

In view of Lemma 4.5, this estimate gives

$$\widetilde{\theta^h}_x \overset{\star}{\rightharpoonup} \theta_x \qquad \text{in } L^\infty(0,T; L^2(\Omega)).$$

Clearly, $\frac{1}{\tau}, \theta \in L^\infty(0,T; L^\infty(\Omega))$, such that

$$\sup_{t \in (0,T)} \int_\Omega |L'(\theta)_x|^2 \, dx = \sup_{t \in (0,T)} \int_\Omega \frac{|\kappa(\theta)|^2}{\tau} |\theta_x|^2 \, dx$$

$$\leq C \|\theta\|_{L^\infty(0,T;L^\infty)} \left\| \frac{1}{\tau} \right\|_{L^\infty(0,T;L^\infty(\Omega))} \|\theta_x\|_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

15

The other terms in (5.4) are treated similarly and hence we conclude (2.11). □

5.2 Proof of Lemma 5.2

Lemma 5.3 *Assume* [A1] *and* [A2]. *Then there is a number* $C > 0$, *independent of* h, *such that*

$$\sup_{t \in (0,T)} \int_\Omega |u^h(x,t)|^2 \, dx + \int_0^T \int_\Omega |u_x^h(x,t)|^2 \, dx dt \ \leq \ C \,. \qquad (5.6)$$

Proof: Applying (3.5) with $v^h = u^h$ and $\mu \equiv \bar{\mu}$, yields

$$\frac{d}{dt} \int_\Omega \frac{|u^h|^2}{2} \, dx + \int_\Omega \frac{\bar{\mu}}{\tau^h} |u_x^h|^2 \, dx = \int_\Omega p(\tau^h, \theta^h) u_x^h \, dx \,. \qquad (5.7)$$

Applying the Cauchy-Schwarz inequality with a suitable parameter ϵ we have

$$\int_\Omega p(\tau^h, \theta^h) u_x^h \, dx \leq \epsilon \int_\Omega \frac{\bar{\mu}}{\tau^h} |u_x^h|^2 \, dx + \frac{\tilde{C}}{\epsilon} \| \frac{1}{\tau^h} \|_{L^\infty(\Omega)} \| \theta^h \|_{L^\infty(\Omega)} \int_\Omega \theta^h \, dx \,.$$

We choose ϵ small enough that the first term can be absorbed on the left-hand side in (5.7). We then apply (3.8) together with (5.1). Integrating in time and using (5.3), yield (5.6) with a suitable C. □

Bound for $\mathcal{A}^h$

Applying (3.5) with the test function $v^h(x,t) := u_t^h(x,t)$, and $\mu \equiv \bar{\mu}$, gives

$$\int_\Omega |u_t^h|^2 \, dx = - \int_\Omega \frac{\bar{\mu} u_x^h u_{tx}^h}{\tau^h} - p^h u_{tx}^h \, dx$$

$$= -\bar{\mu} \frac{d}{dt} \left[\int_\Omega \frac{|u_x^h|^2}{2\tau^h} \, dx \right] + \int_\Omega \frac{\bar{\mu} |u_x^h|^2}{2} \left[\frac{1}{\tau^h} \right]_t + p^h u_{xt}^h \, dx \,,$$

where $p^h := p(\theta^h, \tau^h)$. Integrating in time and using $\tau_t^h = u_x^h$ we obtain

$$\bar{\mu} \int_\Omega \frac{|u_x^h|^2}{2\tau^h} (t) \, dx + \int_0^t \int_\Omega |u_t^h|^2 \, dx ds \qquad (5.8)$$

$$= \int_\Omega \frac{|u_x^h|^2}{2\tau^h} (0) \, dx - \int_0^t \int_\Omega \frac{\bar{\mu} (u_x^h)^3}{2(\tau^h)^2} + p_t^h u_x^h \, dx ds + \int_\Omega p^h u_x^h(s) \, dx \Big|_0^t =: \sum_{i=1}^4 I_i \,,$$

where we have also applied integration by parts to the pressure term. From the requirements on the initial data, we have that $I_1 \leq \tilde{C}$. Next, by adding and subtracting the positive term $|u_x^h|^2 p^h / \tau^h$, we have

$$I_2 = -\frac{1}{2} \int_0^t \int_\Omega \frac{\bar{\mu} (u_x^h)^3}{(\tau^h)^2} \, dx ds \leq \left| \int_0^t \int_\Omega \frac{(u_x^h)^2 \mathcal{F}^h}{\tau^h} \, dx ds \right| \,.$$

Applying Lemma 5.4, with $\phi = \frac{|u_x^h|^2}{|\tau^h|}$, gives

$$I_2 \ \leq \ \tilde{C} \left\{ \epsilon \left[1 + \mathcal{A}^h(t) \right] + \frac{1}{\epsilon} \int_0^t \left[\int_\Omega \frac{|u_x^h|^2}{\tau^h} (x,s) \, dx \right]^2 ds \right\}$$

$$\leq \ \tilde{C} \left\{ \epsilon \left[1 + \mathcal{A}^h(t) \right] + \frac{1}{\epsilon} \int_0^t \mathcal{A}^h(s) \| u_x^h(s) \|_{L^2(\Omega)}^2 \, ds \right\} \,.$$

16

Next, $p^h = \frac{K\theta^h}{\tau^h}$, such that

$$I_3 = K \int_0^t \int_\Omega \frac{|u_x^h|^2 \theta^h}{|\tau^h|^2} - \frac{u_x^h \theta_t^h}{\tau^h} \; dxds \leq \tilde{C} \left[\int_0^t \mathcal{A}^h(s) \|\theta^h(s)\|_{L^\infty(\Omega)} \; ds + \mathcal{D}^h(t)^{\frac{1}{2}} \right],$$

where we have used (5.1), (5.2), and Lemma 5.3. To bound I_4 we use the Cauchy-Schwarz inequality with a parameter and (5.2) together with the requirements on the initial data:

$$I_4 = \int_\Omega p^h u_x^h(s) \; dx \Big|_{s=0}^{s=t} \leq \tilde{C} \left[\epsilon \mathcal{A}^h(t) + \frac{1}{\epsilon} \mathcal{B}^h(t) \right] + \|p^h(0)\|_{L^2(\Omega)} \|u_x(0)\|_{L^2(\Omega)}.$$

Using these bounds in (5.8), taking the supremum over times in $(0, t)$, applying (5.1), and choosing ϵ sufficiently small, we obtain

$$\mathcal{A}^h(t) \leq \tilde{C} \left\{ 1 + \mathcal{B}^h(t) + \mathcal{D}^h(t)^{\frac{1}{2}} + \int_0^t \mathcal{A}^h(t) \left[\|u_x^h\|_{L^2(\Omega)}^2 + \|\theta^h\|_{L^\infty(\Omega)} \right] \; dt \right\}.$$

Applying Grönwall's inequality together with (5.3) and Lemma 5.3 then gives

$$\mathcal{A}^h(t) \leq \tilde{C} \left[1 + \mathcal{B}^h(t) + \mathcal{D}^h(t)^{\frac{1}{2}} \right]. \tag{5.9}$$

Bound for $\mathcal{B}^h$

To bound the $\mathcal{B}^h$ functional we define the test functions

$$\psi^h(x, t) := L'(\theta^h(x, t)),$$

where L is given in **[A1]**. Using ψ^h as test function in (3.6), with $\mu \equiv \bar{\mu}$, integrating in time, and rearranging yield

$$\int_\Omega L(\theta^h(t)) \; dx + \frac{1}{h} \sum_{i=1}^{N-1} \int_0^t G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 \; ds$$

$$= \int_0^t \int_\Omega \mathcal{F}^h u_x^h L'(\theta^h) \; dxds + \int_\Omega L(\theta^h)(0) \; dx =: I_1 + I_2.$$

Using the Cauchy-Schwarz inequality and the lower bounds in (5.1)-(5.2), gives

$$\begin{aligned}
I_1 &\leq \tilde{C} \left\{ \int_0^t \int_\Omega |u_x^h|^2 |\theta^h|^{\beta+1} \; dxds + \int_0^t \int_\Omega |\mathcal{F}^h|^2 |\theta^h|^{\beta+1} \; dxds \right\} \\
&\leq \tilde{C} \left\{ \int_0^t \int_\Omega |u_x^h|^2 |\theta^h|^{\beta+\frac{3}{2}} \; dxds + \int_0^t \int_\Omega |\theta^h|^2 |\theta^h|^{\beta+1} \; dxds \right\} \\
&\leq \tilde{C} \left\{ \left[\sup_{s \in (0,t)} \|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+3} \right]^{\frac{1}{2}} + \int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)} \left[\int_\Omega L(\theta^h) dx \right] \; ds \right\} \\
&\leq \tilde{C} \left\{ 1 + \mathcal{D}^h(t)^{\frac{1}{2}} + \int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)} \mathcal{B}^h(s) \; ds \right\},
\end{aligned}$$

where Lemma 5.5 is applied in the last inequality. The I_2 term is bounded by the requirements on the initial data, whence

$$\mathcal{B}^h(t) \leq \tilde{C} \left\{ 1 + \mathcal{D}^h(t)^{\frac{1}{2}} + \int_0^t \|\theta^h(s)\|_{L^\infty} \mathcal{B}^h(s) \; ds \right\}.$$

An application of Grönwall's inequality, where we use the bound (5.3), yields

$$\mathcal{B}^h(t) \leq \tilde{C} \left[1 + \mathcal{D}^h(t)^{\frac{1}{2}} \right]. \tag{5.10}$$

17

Bound for $\mathcal{D}^h$

To bound $\mathcal{D}^h$ we define the test function $\psi^h \in Q_h(\Omega)$ by

$$\psi^h(x,t) = L'(\theta^h(x,t))_t = \kappa(\theta^h(x,t))\theta_t^h(x,t) \,.$$

Using ψ^h in the temperature scheme (3.6) and integrating in time give

$$\int_0^t \int_\Omega \kappa(\theta^h)|\theta_t^h|^2 \; dxds + \frac{1}{2h} \sum_{i=1}^{N-1} \int_0^t G_i(\tau^h)\big([\![L'(\theta^h)]\!]_i^2\big)_t \; ds$$

$$= \int_0^t \int_\Omega \mathcal{F}^h u_x^h \big(L'(\theta^h)\big)_t \; dxds \,.$$

Integrating by parts in time and rearranging gives

$$\frac{1}{2h} \sum_{i=1}^{N-1} G(\tau^h)[\![L'(\theta^h)]\!]_i^2 + \int_0^t \int_\Omega \kappa(\theta^h)|\theta_t^h|^2 \; dxds$$

$$= \int_0^t \int_\Omega \mathcal{F}^h u_x^h \big(L'(\theta^h)\big)_t dxds + \frac{1}{2h} \sum_{i=1}^{N-1} \Big[G_i(\tau^h)[\![L'(\theta^h)]\!]_i^2 \Big](0)$$

$$+ \frac{1}{2h} \sum_{i=1}^{N-1} \int_0^t \big[G_i(\tau^h)\big]_t [\![L'(\theta^h)]\!]_i^2 \; ds =: J_1 + J_2 + J_3 \,. \tag{5.11}$$

Using the Cauchy-Schwarz inequality with parameter ϵ together with the pointwise bounds (5.1), (5.2) on τ and θ (and the assumption $\beta > 0$), we get

$$J_1 \le \epsilon \int_0^t \int_\Omega \kappa(\theta^h)|\theta_t^h|^2 \; dxds + \frac{\tilde{C}}{\epsilon} \int_0^t \int_\Omega \kappa(\theta^h)|\mathcal{F}^h|^2|u_x^h|^2 \; dxds$$

$$\le \epsilon \mathcal{D}^h(t) + \frac{\tilde{C}}{\epsilon} \int_0^t \|\kappa(\theta^h)\|_{L^\infty} \left[\int_\Omega |\theta^h|^4 + |\mathcal{F}^h|^4 \; dx \right] dt$$

$$\le \epsilon \mathcal{D}^h(t) + \frac{\tilde{C}}{\epsilon} \Big[\sup_{s \in (0,t)} \int_\Omega L(\theta^h) \; dx \Big] \Big[\int_0^t \|\theta^h\|_{L^\infty(\Omega)}^2 \; ds \Big]$$

$$+ \frac{\tilde{C}}{\epsilon} \Big[\sup_{s \in (0,t)} \|\kappa(\theta^h)\|_{L^\infty(\Omega)} \Big]\Big[\sup_{s \in (0,t)} \int_\Omega |\mathcal{F}^h|^2 \; dx \Big]\Big[\int_0^t \|\mathcal{F}^h\|_{L^\infty(\Omega)}^2 \; ds \Big]$$

$$\le \epsilon \mathcal{D}^h(t) + \frac{\tilde{C}}{\epsilon} \mathcal{B}^h(t) \int_0^t \|\theta^h\|_{L^\infty(\Omega)}^2 \; ds$$

$$+ \frac{\tilde{C}}{\epsilon} \Big[\sup_{s \in (0,t)} \|\kappa(\theta^h)\|_{L^\infty(\Omega)} \Big]\Big[\mathcal{A}^h(t) + \mathcal{B}^h(t) \Big]\Big[\int_0^t \|\mathcal{F}^h\|_{L^\infty(\Omega)}^2 \; ds \Big] \,.$$

Applying Lemma 5.6 and the bounds (5.9) and (5.10), we obtain

$$J_1 \le \epsilon \mathcal{D}^h(t) + \frac{\tilde{C}}{\epsilon} \mathcal{B}^h(t) \Big[1 + \mathcal{B}^h(t)^{\frac{1}{2}} \Big]$$

$$+ \frac{\tilde{C}}{\epsilon} \Big[\sup_{s \in (0,t)} \|\kappa(\theta^h)\|_{L^\infty(\Omega)} \Big]\Big[\mathcal{A}^h(t) + \mathcal{B}^h(t) \Big]\Big[1 + \mathcal{A}^h(t)^{\frac{1}{2}} \Big]$$

$$\le \epsilon \mathcal{D}^h(t) + \frac{\tilde{C}}{\epsilon} \Big[1 + \mathcal{D}^h(t)^{\frac{3}{4}} \Big]\Big[1 + \sup_{s \in (0,t)} \|\kappa(\theta^h)\|_{L^\infty(\Omega)} \Big] \,.$$

To bound the last factor we apply Lemma 5.5:

$$\sup_{s \in (0,t)} \|\kappa(\theta^h)\|_{L^\infty(\Omega)} = \tilde{C} \sup_{s \in (0,t)} \Big[\|\theta^h\|_{L^\infty(\Omega)}^{2\beta+3} \Big]^{\frac{\beta}{2\beta+3}} \le \tilde{C} \Big[1 + \mathcal{D}^h(t)^{\frac{\beta}{2\beta+3}} \Big] \,.$$

18

Thus,

$$J_1 \le \epsilon \mathcal{D}^h(t) + \frac{C}{\epsilon} \mathcal{D}^h(t)^{\frac{10\beta+9}{4(2\beta+3)}}, \tag{5.12}$$

where we note that $\frac{10\beta+9}{4(2\beta+3)} < 1$, since $\beta < \frac{3}{2}$. Next, consider J_2: by the requirements on the initial data

$$J_2 = \frac{1}{2h} \sum_{i=1}^{N-1} \left[G_i(\tau_0^h) [\![L'(\theta_0^h)]\!]_i^2 \right] \le \frac{C}{h} \Big[\sum_{i=1}^{N-1} [\![\theta_0^h]\!]_i^2 \Big] \le C \| \theta_{0,x} \|_{L^2(\Omega)}^2. \tag{5.13}$$

To bound J_3 we first observe that for each $1 \le i \le N$,

$$\left(G_i(\tau^h) \right)_t = -\tfrac{1}{2} G_i(\tau^h)^2 \left[u_x^h|_{E_i} + u_x^h|_{E_{i+1}} \right].$$

Writing $u_x^h = \frac{\tau^h}{\mu}(\mathcal{F}^h + p^h)$ and using the positivity of θ^h together with both bounds in (5.1), we get

$$|G_i(\tau^h)_t| \le \tilde{C} G_i(\tau^h) \| \mathcal{F}^h \|_{L^\infty(\Omega)}, \qquad \text{where } \tilde{C} > 0,$$

and thus:

$$
\begin{aligned}
J_3 &= \frac{1}{2h} \sum_{i=1}^{N-1} \int_0^t \left(G_i(\tau^h) \right)_t [\![L'(\theta^h)]\!]_i^2 \, ds \\
&\le \frac{\tilde{C}}{h} \int_0^t \left[\sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 \right] \| \mathcal{F}^h \|_{L^\infty} \, ds \\
&\le \tilde{C} \Big[\sup_{s \in (0,t)} \sum_{i=1}^{N-1} \frac{G_i(\tau^h)}{2h} [\![L'(\theta^h)]\!]_i^2 \Big]^{\frac{1}{2}} \int_0^t \| \mathcal{F}^h \|_{L^\infty} \Big[\sum_{i=1}^{N-1} \frac{G_i(\tau^h)}{h} [\![L'(\theta^h)]\!]_i^2 \Big]^{\frac{1}{2}} ds \\
&\le \tilde{C} \mathcal{D}^h(t)^{\frac{1}{2}} \cdot \Big[\int_0^t \| \mathcal{F}^h \|_{L^\infty(\Omega)}^2 \, ds \Big]^{\frac{1}{2}} \cdot \Big[\frac{1}{h} \sum_{i=1}^{N-1} \int_0^t G_i(\tau^h) [\![L'(\theta^h)]\!]_i^2 \, ds \Big]^{\frac{1}{2}} \\
&\le \tilde{C} \mathcal{D}^h(t)^{\frac{1}{2}} \left[1 + \mathcal{A}^h(t)^{\frac{1}{2}} \right]^{\frac{1}{2}} \mathcal{B}^h(t)^{\frac{1}{2}} \le \tilde{C} \left[1 + \mathcal{D}^h(t)^{\frac{7}{8}} \right], \tag{5.14}
\end{aligned}
$$

where we have used Lemma 5.6 together with the previously derived bounds on $\mathcal{A}^h$ and $\mathcal{B}^h$. Substituting (5.12), (5.13), and (5.14) into (5.11) we obtain

$$\mathcal{D}^h(t) \le \tilde{C} \left[1 + \mathcal{D}^h(t)^\delta \right], \qquad \text{with } \delta := \max \left\{ \tfrac{7}{8}, \tfrac{10\beta+9}{8\beta+6} \right\} < 1,$$

such that

$$\mathcal{D}^h(t) \le \tilde{C}.$$

Recalling (5.9), (5.10) we conclude that there is a $C > 0$, depending on T, the systems parameters, and initial data, but independent of h, and such that

$$\mathcal{A}^h(t) + \mathcal{B}^h(t) + \mathcal{D}^h(t) \le C, \qquad \text{for all } t \in [0, T].$$

This concludes the proof of Lemma 5.2. $\qquad\qquad\square$

Technical lemmas used in the proof

Lemma 5.4 *Assume [A1]-[A2]. Then there is a C independent of h such that*

$$
\left| \int_0^t \int_\Omega \mathcal{F}^h(x,s) \phi(x,s) \, dx ds \right| \le
$$
$$
C \left\{ \epsilon \left[1 + \mathcal{A}^h(t) \right] + \frac{1}{\epsilon} \int_0^t \left[\int_\Omega |\phi(x,s)| \, dx \right]^2 ds \right\},
$$

for all $\epsilon > 0$, $t \in (0,T)$ and $\phi \in L^2(0,T; L^1(\Omega))$.

Proof: For each $t \in (0,T)$ define the test functions v^h by

$$v^h(x,t) = \int_0^x [\Pi_h^Q \phi](y,t) - \left[\frac{1}{|\Omega|} \int_\Omega \phi(z,t) \; dz \right] \; dy \,.$$

Using v^h as test function in (3.5), rearranging and integrating in time, give

$$\left| \int_0^t \int_\Omega \mathcal{F}^h \phi \; dxds \right| = \left| \int_0^t \int_\Omega u_t^h v^h dxds - \frac{1}{|\Omega|} \int_0^t \left[\int_\Omega \phi \; dx \right] \left[\int_\Omega \mathcal{F}^h dx \right] ds \right|$$

$$\leq \int_0^t \left[\int_\Omega |u_t^h| \; dx \right] \left[\int_\Omega |\phi| \; dx \right] ds + C \int_0^t \left[\int_\Omega |\phi| \; dx \right] \left[\int_\Omega \left(|u_x^h| + \theta^h \right) \; dx \right] ds$$

$$\leq \epsilon \mathcal{A}^h(t) + \frac{C}{\epsilon} \int_0^t \left[\int_\Omega |\phi| \; dx \right]^2 ds + \epsilon \int_0^t \int_\Omega |u_x^h|^2 \; dxds + C\epsilon t \,,$$

where we have applied the Cauchy-Schwarz inequality with parameter ϵ, together with the bound (3.8). Applying the estimate (5.6) concludes the proof. $\qquad \square$

Lemma 5.5 *Assume* **[A1]**-**[A2]**. *Then there is a C independent of h such that*

$$\left[\sup_{s \in (0,t)} \|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+3} \right]^\alpha \leq C \left[1 + \mathcal{D}^h(t)^\alpha \right],$$

for all $0 < \alpha < 1$.

Proof: Fix any $s \in (0,t)$. Since $\theta^h = \theta^h(\cdot,s)$ is piecewise constant, we have

$$\theta^h L'(\theta^h)^2 \big|_{E_i} = \theta^h \big|_{E_i} \left[L'(\theta^h)^2 \big|_{E_j} \pm \sum_{k=j}^{i-1} [\![L'(\theta^h)^2]\!]_k \right]$$

$$= \theta^h \big|_{E_i} L'(\theta^h)^2 |_{E_j} \pm \theta^h \big|_{E_i} \sum_{k=j}^{i-1} [\![L'(\theta^h)]\!]_k \left[L'(\theta^h) \big|_{E_k} + L'(\theta^h) \big|_{E_{k+1}} \right].$$

Multiplying through by h, summing over $j = 1, \ldots, N$, taking the maximum over i, and applying the Cauchy–Schwartz inequality, we deduce

$$\|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+3} \leq \|\theta^h(s)\|_{L^\infty(\Omega)} \int_\Omega L'(\theta^h(s))^2 \; dx$$

$$+ 2\|\theta^h(s)\|_{L^\infty(\Omega)}^{\frac{2\beta+3}{2}} \left[\int_\Omega \theta^h(s) \; dx \right]^{\frac{1}{2}} \left[\sum_{k=1}^{N-1} \frac{1}{h} [\![L'(\theta^h(s))]\!]_k^2 \right]^{\frac{1}{2}}$$

$$\leq \|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+2} + \tilde{C} \|\theta^h(s)\|_{L^\infty(\Omega)}^{\frac{2\beta+3}{2}} \left[\sum_{k=1}^{N-1} \frac{1}{h} [\![L'(\theta^h(s))]\!]_k^2 \right]^{\frac{1}{2}}$$

$$\leq \epsilon \|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+3} + \tilde{C}(\epsilon) \left[1 + \sum_{k=1}^{N-1} \frac{1}{h} [\![L'(\theta^h(s))]\!]_k^2 \right],$$

where we have used the energy bound (3.8). We choose ϵ suitably small, absorb the first term on the right-hand side into the left-hand side, and take the supremum over time. Recalling the definition of $\mathcal{D}^h(t)$, using the lower bound $(5.1)_1$, and taking the αth power of both sides, then yields

$$\left[\sup_{s \in (0,t)} \|\theta^h(s)\|_{L^\infty(\Omega)}^{2\beta+3} \right]^\alpha \leq \tilde{C} \left[1 + \sup_{s \in (0,t)} \sum_{k=1}^{N-1} \frac{1}{h} [\![L'(\theta^h)(s)]\!]_k^2 \right]^\alpha \leq C \left[1 + \mathcal{D}^h(t)^\alpha \right].$$

$$\square$$

Lemma 5.6 *Assume* **[A1]**-**[A2]**. *Then there is a C independent of h such that*

$$\int_0^t \|\mathcal{F}^h(s)\|_{L^\infty(\Omega)}^2 \, ds \le C\left[1 + \mathcal{A}^h(t)^{\frac{1}{2}}\right]. \tag{5.15}$$

and

$$\int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)}^2 \, ds \le C\left[1 + \mathcal{B}^h(t)^{\frac{1}{2}}\right], \tag{5.16}$$

Proof: Consider the ith element E_i of the mesh $\mathbb{E}_h$ and let $\mathcal{F}_i^h$ denote the constant value of $\mathcal{F}^h$ on this element. We have

$$|\mathcal{F}_i^h|^2 = |\mathcal{F}_j^h|^2 \pm \sum_{k=j}^i [\![|\mathcal{F}^h|^2]\!]_k \, ,$$

and we proceed to multiply by $\sigma^{\frac{1}{2}} h$, integrate in time, and sum over j. Making use of the lower bound on τ^h, (3.8), (5.3) and Lemma 5.3 to bound $\int\int |\mathcal{F}^h|^2 \, dxds$ we obtain

$$\int_0^t |\mathcal{F}_i^h|^2 \, ds \le \tilde{C}\left\{ \int_0^t \int_\Omega |\mathcal{F}^h|^2 \, dxds + \int_0^t \sum_{k=1}^{N-1} [\![\mathcal{F}^h]\!]_k (\mathcal{F}_k^h + \mathcal{F}_{k+1}^h) \, ds \right\}$$

$$\le \tilde{C}\left\{ 1 + \left[\int_0^t \int_\Omega |\mathcal{F}^h|^2 \, dxds\right]^{\frac{1}{2}} \left[\int_0^t \sum_{k=1}^{N-1} h\frac{[\![\mathcal{F}^h]\!]_k^2}{h^2} \, ds\right]^{\frac{1}{2}} \right\}$$

$$\le \tilde{C}\left\{ 1 + \left[\int_0^t \sum_{k=1}^{N-1} h\frac{[\![\mathcal{F}^h]\!]_k^2}{h^2} \, ds\right]^{\frac{1}{2}} \right\}, \tag{5.17}$$

where we have used the Cauchy-Schwarz inequality. To bound the remaining term, let x_j be an arbitrary interior node and let $v^h \in V_h^0(\Omega)$ be such that $v^h(x_k) = 1$ and $v^h(x_l) = 0$, $\forall l \neq k$. Using v^h as a test function in (3.5) we have

$$[\![\mathcal{F}^h]\!]_k = -\int_{E_{k+1}} \mathcal{F}_{k+1}^h v_x^h \, dy + \int_{E_k} \mathcal{F}_k^h v_x^h \, dy = \int_{E_k \cup E_{k+1}} u_t^h v^h \, dx \, .$$

such that

$$\frac{|[\![\mathcal{F}^h]\!]_k|}{h} \le \left(\Pi_h^Q |u_t^h|\right)\big|_{E_k} + \left(\Pi_h^Q |u_t^h|\right)\big|_{E_{k+1}} \, .$$

Since k is arbitrary it follows that

$$\sum_{k=1}^{N-1} h\frac{[\![\mathcal{F}^h]\!]_k^2}{h^2} \le \tilde{C}\int_\Omega \left(\Pi_h^Q |u_t^h|\right)^2 \, dx \le \tilde{C}\int_\Omega |u_t^h|^2 \, dx \, .$$

Using this in (5.17) yields

$$\int_0^t |\mathcal{F}_i^h|^2 \, ds \le \tilde{C}\left[1 + \mathcal{A}^h(t)^{\frac{1}{2}}\right],$$

and (5.15) follows. To establish (5.16) we argue similarly to get

$$|\theta_i^h|^2 \le \left\{ \int_\Omega |\theta^h|^2 \, dx + \sum_{k=1}^{N-1} \left(\theta_k^h + \theta_{k+1}^h\right)|[\![\theta^h]\!]_k| \right\} \, .$$

Using (3.8) and (5.3) to bound $\int\int |\theta^h|^2 \, dxds$, together with $(5.1)_1$, we obtain

$$\int_0^t |\theta_i^h|^2 \, ds \leq \tilde{C} \left\{ 1 + \int_0^t \sum_{k=1}^{N-1} (\theta_k^h + \theta_{k+1}^h) |[\![\theta^h]\!]_k| \, ds \right\}$$

$$\leq \tilde{C} \left\{ 1 + \left[\int_0^t \int_\Omega |\theta^h|^2 \, dxds \right]^{\frac{1}{2}} \left[\int_0^t \frac{1}{h} \sum_{k=1}^{N-1} [\![\theta^h]\!]_k^2 \, ds \right]^{\frac{1}{2}} \right\}$$

$$\leq \tilde{C} \left\{ 1 + \left[\frac{1}{h} \sum_{k=1}^{N-1} \int_0^t G_k(\tau^h) [\![L'(\theta^h)]\!]_k^2 \, ds \right]^{\frac{1}{2}} \right\} \leq \tilde{C} \left[1 + \mathcal{B}^h(t)^{\frac{1}{2}} \right],$$

where we have used that $[\![L'(\theta^h)]\!]_k = \kappa(\theta^*) [\![\theta^h]\!]_k \geq \tilde{C} [\![\theta^h]\!]_k$, (for a θ^* between θ_k and θ_{k+1}) due to the lower bound (5.2). $\qquad \square$

6 Proof of Lemma 5.1

Throughout this section the assumptions [A1] and [A2] in Section 5 are in force, and we fix an arbitrary time $T > 0$. To simplify the notation we set

$$\omega^h(x,t) := \int_0^t \mathcal{F}^h(x,s) \, ds + \Pi_h^Q \left[\int_0^x u_0^h(y) \, dy \right].$$

In what follows $\tilde{C}$, $C_1, \ldots$ denote positive numbers that depend on T, $|\Omega|$, the initial data, and the bounds (2.9), but that are independent of h.

Lemma 6.1 *There is a number $0 < C_1 < \infty$ such that*

$$|\omega^h(x,t)| \leq C_1, \qquad \text{for all } (x,t) \in \Omega \times [0,T].$$

Proof: For fixed $t > 0$ and $v^h \in V_h^0(\Omega)$, integrating (3.5) in time gives

$$\int_\Omega \omega^h v_x^h \, dx = - \int_\Omega u^h v^h \, dx. \qquad (6.1)$$

Using $v^h(x) = u^h(x,t)$ then yields

$$\int_\Omega \omega^h u_x^h \, dx = - \int_\Omega |u^h|^2 \, dx.$$

Using this together with (3.4) with $\phi^h(x) = \omega^h(x,t)$, gives[1]

$$\frac{d}{dt} \int_\Omega \omega^h \tau^h \, dx = \int_\Omega \mathcal{F}^h \tau^h \, dx + \int_\Omega \omega^h u_x^h \, dx = - \int_\Omega K\theta^h + |u^h|^2 \, dx.$$

According to (3.8) we thus have, for each $t \geq 0$, that

$$\tilde{C} - t(K+2)\mathcal{E}(\theta_0^h, u_0^h) \leq \left(\int_\Omega \tau^h \omega^h dx \right)(t) \leq \int_\Omega \omega^h(x,0)\tau_0^h(x) \, dx =: \tilde{C}. \quad (6.2)$$

Next, by conservation of mass (3.7) there are maps $t \mapsto z^\pm(t) \in \Omega$ such that

$$\omega^h(z^-(t),t) \leq \left(\int_\Omega \tau^h \omega^h dx \right)(t) \leq \omega^h(z^+(t),t), \quad \forall t. \qquad (6.3)$$

---

[1]At this point the assumption of constant viscosity is used in an essential manner.

Next, fix t, $h > 0$, and $y \in \Omega$ arbitrary. Let $E^- \in \mathbb{E}_h$ be the element containing $z^-(t)$, let E^y be the element containing y, and define the interval $S^y \subseteq \Omega$ by

$$S^y := \left[\min\{z^-(t), y\}, \max\{z^-(t), y\}\right] \cup E^- \cup E^y .$$

Then fix a $v^h \in V_h^0(\Omega)$ by requiring that for each x_i, $i \in \{0, 1, \ldots, N\}$,

$$v^h(x_i) = \begin{cases} 1 & \text{if } x_i \in S^y, \\ 0 & \text{otherwise .} \end{cases}$$

Since ω^h is piecewise constant we have

$$\omega^h(y, t) - \omega^h(z^-(t), t) = \pm \sum_{\{i : x_i \in S^y\}} \llbracket \omega^h \rrbracket_i$$

$$= \pm \sum_{i=1}^{N-1} \llbracket \omega^h \rrbracket_i v^h(x_i) = \mp \int_\Omega \omega^h v_x^h \, dx = \pm \int_\Omega u^h v^h \, dx,$$

where the last equality follows by (6.1). Rearranging, applying Hölder and the energy estimate (3.8), using (6.3) together with (6.2), and finally taking the supremum in y, give

$$\sup_{y \in \Omega} \omega^h(y, t) \leq \tilde{C} .$$

A similar argument using $z^+(t)$ yields $\inf_{y \in \Omega} \omega^h(y, t) \geq -\tilde{C}$. $\qquad \square$

Lemma 6.2 *There exists $C_2 > 0$ such that*

$$C_2^{-1} \leq \|\tau^h(t)\|_{L^\infty(\Omega)} \leq C_2 \left[1 + \int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)} \, ds\right] \qquad \text{for } t \in [0, T] .$$

Proof: Since $\tau_t^h(\cdot, t)$, $u_x^h(\cdot, t) \in Q_h(\Omega)$ at each time, (3.4) gives $\tau_t^h = u_x^h$. Thus, with $A := (\tau^h)^{-1} \exp(\bar{\mu}^{-1} \omega^h)$ and $B := A^{-1}$, we get

$$A_t = -\frac{p(\theta^h, \tau^h)}{\bar{\mu}} A \leq 0 , \qquad B_t = \frac{p(\theta^h, \tau^h)}{\bar{\mu}} B = \frac{K}{\bar{\mu}} \theta^h \exp(-\frac{\omega^h}{\bar{\mu}}) .$$

Integration in time, together with (2.9) and Lemma 6.1, give

$$A(x, t) \leq A_0(x) \leq \tilde{C} \quad \Rightarrow \quad \tau^h(x, t) \geq C_2^{-1} ,$$

and

$$B(x, t) \leq B_0(x) + \tilde{C} \int_0^t \|\theta^h(s)\|_\infty ds \quad \Rightarrow \quad \tau^h(x, t) \leq C_2 \left[1 + \int_0^t \|\theta^h(s)\|_\infty \, ds\right] .$$

$\square$

Lemma 6.3 *There exists $C_3 > 0$ such that*

$$\left\|\frac{1}{\theta^h(t)}\right\|_{L^\infty(\Omega)} \leq C_3 \qquad \text{for } t \in [0, T] . \tag{6.4}$$

Proof: Apply (3.6) with $\psi^h = -M(\theta^h)^{-M-1}$, where $M > 0$. A calculation (completing the square in u_x^h) shows that

$$\frac{d}{dt} \int_\Omega (\theta^h)^{-M} \, dx - \frac{M}{h} \sum_{i=1}^{N-1} G_i(\tau^h) [\![L'(\theta^h)]\!]_i \left[\!\!\left[\frac{1}{(\theta^h)^{M+1}} \right]\!\!\right]_i \leq \int_\Omega \frac{MK^2 dx}{4\bar{\mu} \tau^h (\theta^h)^{M-1}} \, .$$

Since $L' = \kappa > 0$ and $M > 0$ the sum on the left is negative, and Hölder gives

$$\frac{d}{dt} \left[\left\| \frac{1}{\theta^h(t)} \right\|_M^M \right] \leq \frac{MK^2}{4\bar{\mu}} \int_\Omega \frac{dx}{\tau^h (\theta^h)^{M-1}} \leq \tilde{C} M \left\| \frac{1}{\tau^h(t)} \right\|_M \left\| \frac{1}{\theta^h(t)} \right\|_M^{M-1} \, ,$$

where $\| \cdot \|_M$ denotes the $L^M(\Omega)$-norm. Applying Lemma 6.2 we obtain

$$\frac{d}{dt} \left[\left\| \frac{1}{\theta^h(t)} \right\|_M \right] \leq \tilde{C} \left\| \frac{1}{\tau^h(t)} \right\|_M \leq \tilde{C} \, .$$

Integrating in time, applying $(2.9)_2$, and sending $M \uparrow \infty$, yield (6.4). $\qquad\square$

Lemma 6.4 *There exists $C_4 > 0$ such that*

$$\int_0^T \|\theta^h(s)\|_\infty \, ds \leq C_4 \, . \tag{6.5}$$

Proof: Fix $x, y \in \Omega$. As $\theta^h(\cdot, t) \in Q_h(\Omega)$, Lemma 6.3 yields (suppressing t)

$$\theta^h(x)^{\frac{1}{2}} = \theta^h(y)\theta^h(x)^{-\frac{1}{2}} + \sum_{i=j}^{k-1} \frac{[\![\theta^h]\!]_i}{\theta^h(x)^{\frac{1}{2}}} \leq \tilde{C}\theta^h(y) + \sum_{i=1}^{N-1} \frac{|[\![\theta^h]\!]_i|}{\theta^h(x)^{\frac{1}{2}}} \, ,$$

where $x \in E_k$ and $y \in E_j$. Integrating in y, using (3.8), and choosing x such that $\|\theta^h(t)\|_{L^\infty(\Omega)} = \theta^h(x, t)$, yield

$$\|\theta^h(t)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \leq \tilde{C} \left[1 + \sum_{i=1}^{N-1} \frac{|[\![\theta^h]\!]_i|}{\|\theta^h(t)\|_{L^\infty(\Omega)}^{\frac{1}{2}}} \right] \leq \tilde{C} \left[1 + \sum_{i=1}^{N-1} \frac{|[\![\theta^h]\!]_i|}{(\theta_{i*}^h)^{\frac{1}{2}}} \right] \, ,$$

where θ_{i*}^h is as in (3.9). Multiplying and dividing by suitable terms in the latter sum, using $\kappa(\theta) \propto \theta^\beta$, applying Cauchy-Schwarz, and squaring, give

$$\|\theta^h(t)\|_{L^\infty(\Omega)} \leq \tilde{C} \left\{ 1 + \left[\sum_{i=1}^{N-1} \frac{G_i(\tau^h)\kappa(\theta_{i\dagger}^h)}{h(\theta_{i*}^h)^2} [\![\theta^h]\!]_i^2 \right] \left[\sum_{i=1}^{N-1} \frac{h\theta_{i*}^h}{G_i(\tau^h)(\theta_{i\dagger}^h)^\beta} \right] \right\} \, , \tag{6.6}$$

where $\theta_{i\dagger}^h$ is as in (3.9). Let the two inner sums on the right-hand side of (6.6) be $\mathfrak{A}(t)$ and $\mathfrak{B}(t)$, respectively. To bound $\mathfrak{B}(t)$ we recall that $G_i(\tau^h) = 2(\tau^h|_{E_i} + \tau^h|_{E_{i+1}})^{-1}$, and use that $\theta_{i*}^h \leq \theta^h|_{E_i} + \theta^h|_{E_{i+1}}$, while $\theta_{i\dagger}^h$ is between $\theta^h|_{E_i}$ and $\theta^h|_{E_{i+1}}$. Lemma 6.3, (3.8), and Lemma 6.2, then give

$$\mathfrak{B}(t) \leq \tilde{C} \|\tau^h(t)\|_{L^\infty(\Omega)} \leq \tilde{C} \left[1 + \int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)} \, ds \right] \, .$$

Squaring both sides in (6.6) thus gives

$$\|\theta^h(t)\|_{L^\infty(\Omega)} \leq \tilde{C} \left[1 + \mathfrak{A}(t) \cdot \left[1 + \int_0^t \|\theta^h(s)\|_{L^\infty(\Omega)} \, ds \right] \right] \, .$$

Applying the Grönwall inequality and recalling, by (3.10), that $\int_0^t \mathfrak{A}(s) \, ds \leq C_0$, we obtain (6.5). $\qquad\square$

Combing Lemma 6.1 - Lemma 6.4 completes the proof of Lemma 5.1.

24

References

[1] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR **2424078 (2009e:**46025)

[2] James H. Bramble, Joseph E. Pasciak, and Olaf Steinbach, *On the stability of the L^2 projection in $H^1(\Omega)$*, Math. Comp. **71** (2002), no. 237, 147–156 (electronic). MR **1862992 (2002h:**65175)

[3] Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR **2373954 (2008m:**65001)

[4] Didier Bresch and Benoît Desjardins, *On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids*, J. Math. Pures Appl. (9) **87** (2007), no. 1, 57–90 (English, with English and French summaries). MR **2297248 (2008h:**35277)

[5] Sydney Chapman and T. G. Cowling, *The mathematical theory of nonuniform gases*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990. An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases; In co-operation with D. Burnett; With a foreword by Carlo Cercignani. MR **1148892 (92k:**82001)

[6] Gui-Qiang Chen, David Hoff, and Konstantina Trivisa, *Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data*, Comm. Partial Differential Equations **25** (2000), no. 11-12, 2233–2257. MR **1789926 (2001h:**35141)

[7] C. M. Dafermos, *Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity*, SIAM J. Math. Anal. **13** (1982), no. 3, 397–408. MR **653464 (83e:**35104)

[8] C. M. Dafermos and L. Hsiao, *Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity*, Nonlinear Anal. **6** (1982), no. 5, 435–454. MR **661710 (83g:**35059)

[9] Eduard Feireisl, *Dynamics of Viscous Compressible Fluids*, Vol. 26, Oxford University Press, 2004. MR **2040667 (2005i:**76092)

[10] _____, *Mathematical theory of compressible, viscous, and heat conducting fluids*, Comp. Math. Appl **53** (2007), no. 3-4, 461–490. MR **2323704 (2008i:**76160)

[11] David Hoff, *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data*, J. Differential Equations **95** (1992), no. 1, 33–74. MR **1142276 (93f:**35178)

[12] Ja. I. Kanel′, *A model system of equations for the one-dimensional motion of a gas*, Differencial′nye Uravnenija **4** (1968), 721–734 (Russian). MR 0227619 (37 #3203)

[13] Shuichi Kawashima and Mari Okada, *Smooth global solutions for the one-dimensional equations in magnetohydrodynamics*, Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), no. 9, 384–387. MR **694940 (85c:**35078)

[14] Bernhard Kawohl, *Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas*, J. Differential Equations **58** (1985), no. 1, 76–103. MR **791841 (86h:**35119)

[15] Alexander V. Kazhykhov, *Sur la solubilité globale des problèmes monodimensionnels aux valeurs initiales-limitées pour les équations d'un gaz visqueux et calorifère*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 5, A317–A320. MR 0428905 (55 #1925)

[16] A. V. Kazhikhov and V. V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, Prikl. Mat. Meh. **41** (1977), no. 2, 282–291 (Russian); English transl., J. Appl. Math. Mech. **41** (1977), no. 2, 273–282. MR 0468593 (57 #8425)

[17] Pierre-Louis Lions, *Mathematical topics in fluid mechanics. Vol. 1*, Oxford Lecture Series in Mathematics and its Applications, vol. 3, The Clarendon Press Oxford University Press, New York, 1996. Incompressible models; Oxford Science Publications. MR **1422251 (98b:**76001)

[18] Tai-Ping Liu, Zhouping Xin, and Tong Yang, *Vacuum states for compressible flow*, Discrete Contin. Dynam. Systems **4** (1998), no. 1, 1–32. MR **1485360 (98k:**76116)

[19] Tao Luo, Zhouping Xin, and Tong Yang, *Interface behavior of compressible Navier-Stokes equations with vacuum*, SIAM J. Math. Anal. **31** (2000), no. 6, 1175–1191 (electronic). MR **1766564 (2001g:**35275)

[20] A. Mellet and A. Vasseur, *Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations*, SIAM J. Math. Anal. **39** (2007/08), no. 4, 1344–1365. MR **2368905 (2009a:**76151)

[21] John Nash, *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. France **90** (1962), 487–497 (French). MR 0149094 (26 #6590)

[22] James Serrin, *Mathematical principles of classical fluid mechanics*, Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), Springer-Verlag, Berlin, 1959, pp. 125–263. MR 0108116 (21 #6836b)

[23] Jacques Simon, *Compact sets in the space $L^p(0, T; B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96. MR **916688 (89c:**46055)

[24] W. G Vincenti and C. H. Kruger, *Introduction to Physical Gas Dynamics*, Cambridge Mathematical Library, Krieger Pub Co, 1975.

[25] Seak-Weng Vong, Tong Yang, and Changjiang Zhu, *Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum. II*, J. Differential Equations **192** (2003), no. 2, 475–501. MR **1990849 (2004d:**35210)

[26] Tong Yang and Huijiang Zhao, *A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity*, J. Differential Equations **184** (2002), no. 1, 163–184. MR **1929151 (2003g:**35233)

[27] Tong Yang and Changjiang Zhu, *Compressible Navier-Stokes equations with degenerate viscosity coefficient and vacuum*, Comm. Math. Phys. **230** (2002), no. 2, 329–363. MR **1936794 (2003g:**35179)

[28] Zel'dovich Ya. B. and Raizer Yu. P., *Physics of shock waves and high-temperature hydrodynamic phenomena*, Dover Publications, 2002.